UNIT-V

LAPLACE TRANSFORMS

Laplace Transform :- Laplace transform is the frequency domain representation of any continuous periodic or aperiodic signal.

Laplace Transform of any continuous signal $x(t)$ is given as

$$\boxed{L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt}$$

Where "s" is a complex variable and is equal to $\boxed{s = \sigma + j\omega}$

Here the operator 'L' is called the laplace transform operator which transforms the time domain function $x(t)$ into frequency domain function $X(s)$.

Region of Convergence :-

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$$

Substituting $s = \sigma + j\omega$ in the above equation

$$= \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[x(t)\, e^{-\sigma t}\right] e^{-j\omega t}\, dt = F\left[x(t)\, e^{-\sigma t}\right]$$

This equation indicates that $X(s)$ is basically the continuous-time Fourier transform of $x(t)e^{-\sigma t}$.

So we can say that

* The Laplace transform of "$x(t)$" is the fourier transform of "$x(t)\, e^{-\sigma t}$"

Existence of laplace Transform :-

1.) $x(t)$ should be continuous or piece-wise continuous in the given closed interval.

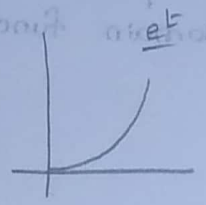
2.) $x(t)e^{-\sigma t}$ must be absolutely integrable.

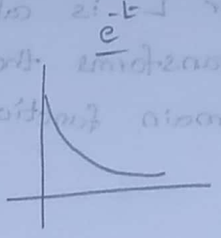
That is, $X(s)$ exists only if $\boxed{\int_{-\infty}^{\infty} |x(t)\, e^{-\sigma t}|\, dt < \infty}$

or only if $\quad \underset{t\to\infty}{Lt}\; e^{-st}\, x(t) = 0$

"The range of σ for which the laplace transform converges is known as the region of convergence (ROC)."

So the functions which are not Fourier transformable may be Laplace transformable.





$e^{-\sigma t}$, if $\sigma = 2 \Rightarrow e^{-2t}$

e^{-t^2}

	e^{-2t}	$e^{-t^2} \to$ drastically decreasing.
$t=3$	e^{-6}	e^{-9}
$t=4$	$e^{-4\sigma}$	e^{-16}

The existance of laplace transform depends on "σ" (Real part of s) ($Re\{s\}$)

1) Find the laplace Transform of $\delta(t)$?

Sol:- $\quad L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$

Here $x(t) = \delta(t)$

$$\delta(t) = \begin{cases} 1 , & t=0 \\ 0 & \text{otherwise} \end{cases}$$

$$X(s) = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt$$

$$= \delta(t) . e^{-st} \Big/_{t=0}$$

$$= \delta(0) . e^{-s(0)}$$

$$= (1)(1)$$

$$\boxed{X(s) = 1}$$

$$\therefore \quad L\{\delta(t)\} = 1$$

(or)

$$\delta(t) \xleftrightarrow{L.T} 1$$

2.) Find the Laplace Transform of $u(t)$

Sol:- The Laplace transform of $x(t)$ is

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

Here $x(t) = u(t)$

$$u(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$X(s) = \int_{-\infty}^{\infty} u(t) e^{-st} dt$$

$$X(s) = L\{u(t)\} = \int_{-\infty}^{\infty} u(t) e^{-st} dt$$

$$= \int_{-\infty}^{0} 0 . e^{-st} dt + \int_{0}^{\infty} 1 . e^{-st} dt$$

$$= 0 + \int_{0}^{\infty} e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_{0}^{\infty}$$

$$= \frac{-1}{s} \left[e^{-\infty(s)} - e^{0} \right]$$

$$= \frac{-1}{s} \left[e^{-(\sigma + j\omega)\infty} - 1 \right]$$

$$= \frac{-1}{s} [0 - 1] = \frac{1}{s}$$

$$\underset{t \to \infty}{Lt} \; e^{-\sigma t} = 0 \; ; \quad \sigma > 0$$

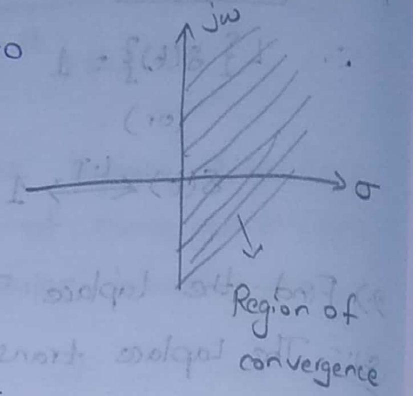
$\therefore \; L\{u(t)\} = \frac{1}{s}, \quad \sigma > 0$

(or) $L\{u(t)\} = \frac{1}{\sigma + j\omega}, \quad \sigma > 0$

(or) $X(s) = \frac{1}{s}, \quad Re\{s\} > 0$

$F\{u(t)\} = \pi\delta(\omega) + \frac{1}{j\omega}$

$L\{u(t)\} = \frac{1}{\sigma + j\omega} = \frac{1}{s}, \quad \sigma > 0$



3.) Find the Laplace Transform of $e^{-at}u(t)$.

The Laplace Transform of $x(t)$ is

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

Here $x(t) = e^{-at}u(t)$

$$X(s) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st} dt$$

$$X(s) = \int_{-\infty}^{0} 0 \cdot e^{-at} e^{-st} dt + \int_{0}^{\infty} (1) e^{-at} e^{-st} dt.$$

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= 0 + \int_{0}^{\infty} e^{-at} e^{-st} dt = \int_{0}^{\infty} e^{-(a+s)t} dt$$

$$= \left[\frac{e^{-(a+s)t}}{-(a+s)} \right]_{0}^{\infty}$$

$$= \frac{e^{-(a+s)\infty}}{-(a+s)} - \frac{e^{-(a+s)0}}{-(a+s)} \; ; \quad s+a > 0$$

$$= 0 + \frac{1}{a+s} = \frac{1}{a+s}.$$

$$= \frac{1}{s+a} \quad Re\{s\} > -a$$

$$\boxed{\therefore \; e^{-at}u(t) \xleftrightarrow{L.T} \frac{1}{s+a} \quad Re\{s\} > -a}$$

(or)

$$e^{-at}u(t) \xleftrightarrow{L.T} \frac{1}{S+a} \quad \sigma > -a$$

$$e^{-at}u(t) \xleftrightarrow{F.T} \frac{1}{a+j\omega} \qquad e^{-at}u(t) \xleftrightarrow{L.T} \frac{1}{\underset{\downarrow\\0}{\boxed{\sigma}}+j\omega+a}$$

If $\sigma = 0$

$$\boxed{L\{e^{-at}u(t)\} = F[e^{-at}u(t)]}$$

$$\frac{1}{a+j\omega} = \frac{1}{a+j\omega}$$

4) Find the Laplace Transform of $-u(-t)$

The laplace Transform of $x(t)$ is

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

Here $x(t) = -u(-t)$

$$X(s) = \int_{-\infty}^{\infty} -u(-t)\, e^{-st}\, dt$$

$$X(s) = \int_{-\infty}^{0} -u(-t)\, e^{-st}\, dt + \int_{0}^{\infty} -u(-t)\, e^{-st}\, dt$$

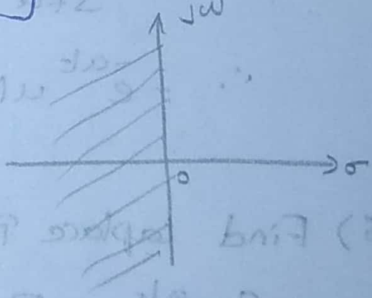
$$u(-t) = \begin{cases} 1 & ; t<0 \\ 0 & ; t>0 \end{cases}$$

$$= \int_{-\infty}^{0} -(1)\, e^{-st}\, dt + \int_{0}^{\infty} 0 \cdot e^{-st}\, dt$$

$$= \left[-\frac{e^{-st}}{-s}\right]_{-\infty}^{0} = \frac{1}{s}\left[e^{-s(0)} - e^{-s(-\infty)}\right]$$

$$= \frac{1}{s}[1-0] \qquad Re\{s\} < 0$$

$$= \frac{1}{s}, \quad Re\{s\} < 0$$



$$\therefore \; -u(-t) \xleftrightarrow{L.T} \frac{1}{s}, \quad Re\{s\} < 0$$

(or)

$$\boxed{-u(-t) \xleftrightarrow{L.T} \frac{1}{s}, \quad \sigma < 0}$$

σ → region of convergence

$$L^{-1}\left[\frac{1}{s}\right], \quad \sigma > 0 : u(t)$$

5) Find laplace Transform of $-e^{-at}u(-t)$

The laplace Transform of $x(t)$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt.$$

Here $x(t) = -e^{-at}u(-t)$

$$X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}\,dt \qquad u(-t) = \begin{cases} 1 & ; t<0 \\ 0 & ; t>0 \end{cases}$$

$$= \int_{-\infty}^{0} -e^{-at}u(-t)e^{-st}\,dt + \int_{0}^{\infty} -e^{-at}u(-t)e^{-st}\,dt$$

$$= \int_{-\infty}^{0} (1) -e^{-(a+s)t}\,dt + 0$$

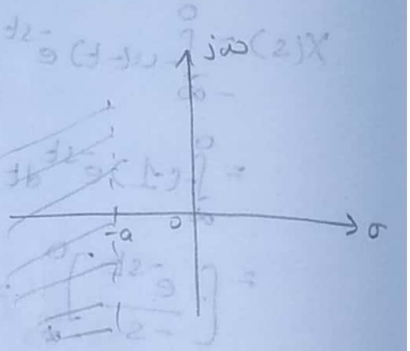
$$= \left[\frac{+e^{-(a+s)t}}{+(a+s)}\right]_{-\infty}^{0} = \frac{e^{-(a+s)0}}{a+s} - \frac{e^{-(a+s)(-\infty)}}{a+s}$$

$$= \frac{1}{a+s}\left[e^{0} - e^{+(a+s)\infty}\right]$$

$$= \frac{1}{a+s}[1-0] \qquad a+s<0$$

$$= \frac{1}{s+a} \qquad s<-a$$

$$= \frac{1}{s+a} \qquad \sigma<-a$$



$$\therefore \quad -e^{-at}u(-t) \xleftrightarrow{L.T} \frac{1}{s+a} \; ; \sigma<-a$$

(6) Find laplace Transform of $e^{-2t}u(t) + e^{3t}u(t)$

$$L[e^{-2t}u(t)] = X(s) = \int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st}\,dt.$$

$$= \int_{-\infty}^{0} e^{-2t}\underbrace{u(t)}_{0}e^{-st}\,dt + \int_{0}^{\infty} u(t)e^{-2t}e^{-st}\,dt.$$

$$= 0 + \int_{0}^{\infty} u(t)e^{-(2+s)t}\,dt$$

$$= \left[\frac{e^{-(2+s)t}}{-(2+s)}\right]_{0}^{\infty} = \frac{e^{-(2+s)\infty} - e^{-(2+s)0}}{-(2+s)}$$

$$= 0 - \frac{1}{-(2+s)} = \frac{1}{2+s} \; ; \; 2+s>0$$

$$= \frac{1}{2+s} \quad ; \quad s > -2$$

$$= \frac{1}{2+s} \quad ; \quad \sigma > -2$$

$$\therefore \boxed{e^{-2t} u(t) \xleftrightarrow{L.T} \frac{1}{2+s} \quad ; \quad \sigma > -2}$$

$$L\left[e^{3t}u(t)\right] = X(s) = \int_{-\infty}^{\infty} e^{3t} u(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{3t} u(t) e^{-st} dt + \int_{0}^{\infty} u(t)\, e^{3t} e^{-st}\, dt$$

$$= 0 + \int_{0}^{\infty} \underbrace{u(t)}_{1}\, e^{3t} e^{-st}\, dt$$

$$= \int_{0}^{\infty} 1 \cdot e^{(3-s)t}\, dt$$

$$= \left[\frac{e^{(3-s)t}}{3-s}\right]_{0}^{\infty} = \frac{1}{3-s}\left[e^{(3-s)\infty} - e^{(3-s)0}\right]$$

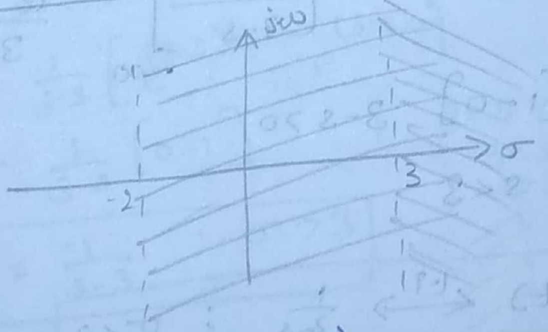
$$= \frac{1}{3-s}\left[0-1\right] \quad ; \quad 3-s<0$$

$$3 < s \Rightarrow s > 3$$

$$= -\frac{1}{3-s} \quad ; \quad s > 3$$

$$\therefore e^{3t} u(t) \xleftrightarrow{L.T} \frac{1}{s-3} \quad ; \quad \sigma > 3$$

$$\therefore \boxed{e^{-2t} u(t) + e^{3t} u(t) \xleftrightarrow{L.T} \frac{1}{s+2} + \frac{1}{s-3} \quad \sigma > 3}$$



(or)

$$e^{-at} u(t) \xleftrightarrow{L.T} \frac{1}{s+a} \quad \sigma > -a$$

$$e^{at} u(t) \xleftrightarrow{L.T} \frac{1}{s-a} \quad \sigma > a$$

$$e^{-2t} u(t) \xleftrightarrow{L.T} \frac{1}{s+2} \quad \sigma > -2$$

$$e^{3t} u(t) \xleftrightarrow{L.T} \frac{1}{s-3} \quad \sigma > 3$$

$$\therefore e^{-2t} u(t) + e^{3t} u(t) \xleftrightarrow{L.T} \frac{1}{s+2} + \frac{1}{s-3} \quad \sigma > 3$$

7) Find $L\{e^{-2t}u(t) + e^{3t}u(-t)\}$

The laplace Transform of $x(t)$ is

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

Here $x(t) = -u(-t)$

$$X(s) = \int_{-\infty}^{\infty} -u(-t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} (1)\, e^{-(2+s)t}\, dt$$

$$= \left[\frac{e^{-(2+s)t}}{-(2+s)}\right]_0^{\infty} = \frac{1}{2+s}\left[-e^{-(2+s)\infty} - (-e^{-(2+s)0})\right]$$

$$= \frac{1}{s+2}[0+1] \qquad 2+s>0 \Rightarrow \frac{1}{s+2} \qquad s>-2$$

$$\therefore \boxed{e^{-2t}u(t) \xleftrightarrow{L.T} \frac{1}{s+2} \;;\; \sigma > -2}$$

(ii) $L[e^{3t}u(-t)] = X(s) = \int_{-\infty}^{\infty} e^{3t}u(-t)\, e^{-st}\, dt$

$$X(s) = \int_{-\infty}^{0} u(-t)\, e^{(3-s)t}\, dt + \int_{0}^{\infty} u(t)\, e^{(3-s)t}\, dt.$$

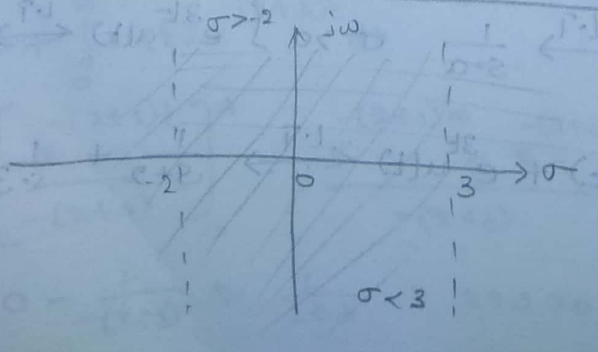
$$= \int_{-\infty}^{0} (1)\, e^{(3-s)t}\, dt = \left[\frac{e^{(3-s)t}}{3-s}\right]_{-\infty}^{0} = \frac{e^{-(3-s)0}}{3-s} - \frac{e^{(3-s)\infty}}{3-s}$$

$$= \frac{1}{3-s}[1-0] \qquad 3-s>0$$

$$= \frac{1}{3-s} \qquad s<3$$

$$\therefore \boxed{e^{3t}u(-t) \xleftrightarrow{L.T} \frac{1}{3-s} \;;\; \sigma<3}$$

$$\therefore \boxed{e^{-2t}u(t) + e^{3t}u(-t) \xleftrightarrow{L.T} \frac{1}{s+2} + \frac{1}{3-s} \;;\; -2<\sigma<3}$$



8) Find $L[e^{-2t}u(-t) + e^{3t}u(t)]$

(i) $L[e^{-2t}u(-t)] = X(s) = \int_{-\infty}^{\infty} e^{-2t}u(-t)\,e^{-st}\,dt.$

$$X(s) = \int_{-\infty}^{0} \underbrace{u(-t)}_{1}\, e^{-(2+s)t}\,dt + \int_{0}^{\infty} \underbrace{u(t)}_{0}\, e^{-(2+s)t}\,dt.$$

$$= \int_{-\infty}^{0} 1\, e^{-(2+s)t}\,dt = \left[\frac{e^{-(2+s)t}}{-(2+s)}\right]_{-\infty}^{0}$$

$$= -\frac{1}{(2+s)}\left[e^{-(2+s)0} - e^{-(2+s)\infty}\right]$$

$$= -\frac{1}{(2+s)}\left[1-0\right]$$

$s+2 < 0$

$s < -2$

$$= \frac{-1}{2+s} \; ; \; \sigma < -2$$

$$\therefore \boxed{e^{-2t}u(-t) \xleftrightarrow{L.T} -\frac{1}{2+s} \; ; \; \sigma < -2}$$

(ii) $L[e^{3t}u(t)] = X(s) = \int_{-\infty}^{\infty} e^{3t}u(t)\,e^{-st}\,dt$

$$X(s) = \int_{-\infty}^{0} e^{3t}e^{-st}\underbrace{u(t)}_{0}\,dt + \int_{0}^{\infty} e^{(3-s)t}\underbrace{u(t)}_{1}\,dt$$

$$= \int_{0}^{\infty} (1)\, e^{(3-s)t}\,dt = \left[\frac{e^{(3-s)t}}{3-s}\right]_{0}^{\infty}$$

$$= \frac{1}{3-s}\left[e^{(3-s)\infty} - e^{(3-s)0}\right]$$

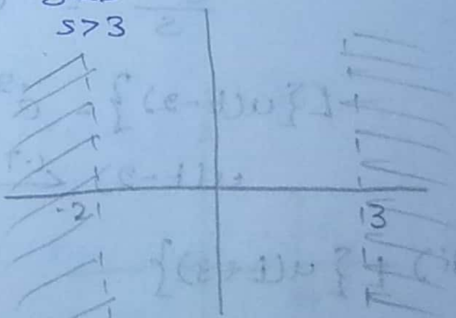
$$= \frac{1}{3-3}\left[0-1\right]$$

; $3-s < 0$

$3 < s$

$s > 3$

$$\boxed{X(s) = \frac{1}{s-3} \; ; \; \sigma > 3}$$



$\sigma < -2$ $\quad s > 3$

∴ No convergence

∴ The laplace Transform does not exists.

$\infty \to <$

$-\infty \to >$

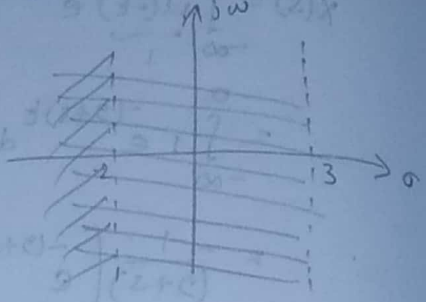
9.) $L\left[e^{-2t}u(-t) + e^{3t}u(-t)\right]$

$e^{-2t}u(-t) \xleftrightarrow{L.T} -\frac{1}{s+2}$; $\sigma < -2$

Prob. 8

$e^{3t}u(-t) \xleftrightarrow{L.T} \frac{1}{3-s}$; $\sigma < 3$

Prob-7



$\therefore L\left[e^{-2t}u(-t) + e^{3t}u(-t)\right] \xleftrightarrow{L.T} -\frac{1}{s+2} + \frac{1}{3-s}$; $\sigma < -2$

10.) Find the laplace Transform of (i) $u(t-2)$

$$X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt.$$

$x(t) = u(t-2)$

$$= \int_{-\infty}^{+\infty} u(t-2)\, e^{-st}\, dt.$$

$u(t-2) = \begin{cases} 1, & t \geq 2 \\ 0 & \text{otherwise} \end{cases}$

$$= \int_{2}^{\infty} 1.\, e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_2^{\infty} = -\frac{1}{s}\left[\underset{=0,\ s>0}{e^{-s(\infty)}} - e^{-s(2)}\right]$$

$$= -\frac{1}{s}\left[0 - e^{-2s}\right], \quad \sigma > 0$$

$$= \frac{e^{-2s}}{s} = e^{-2s}.\frac{1}{s}, \quad \sigma > 0$$

$L\{u(t-2)\} = e^{-2s}.\frac{1}{s}$, $\sigma > 0$

$u(t-2) \xleftrightarrow{L.T} e^{-2s}.\frac{1}{s}$, $\sigma > 0$

(ii) $L\{u(t+3)\}$

$$X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$$

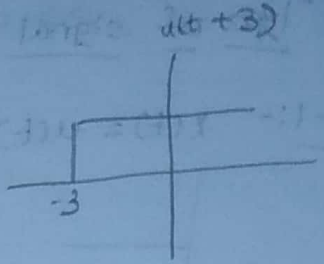
$$= \int_{-\infty}^{\infty} u(t+3)\, e^{-st}\, dt$$

$u(t+3) = \begin{cases} 1 & ; t \geq -3 \\ 0 & , \text{otherwise} \end{cases}$

$$= \int_{-3}^{\infty} 1.\, e^{-st}\, dt.$$

$$= \left[\frac{e^{-st}}{-s}\right]_{-3}^{\infty} = \frac{-1}{s}\left[\underset{=0,\ s>0}{e^{-s(\infty)}} - e^{-s(-3)}\right]$$

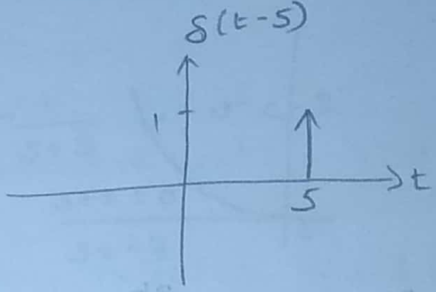
$$= -\frac{1}{s}\left[0 - e^{3s}\right] = \frac{e^{3s}}{s}, \quad \sigma > 0$$



$$L\{u(t+3)\} = e^{3s}\cdot\frac{1}{s} \quad \sigma > 0$$

$$u(t+3) \xleftrightarrow{L.T} e^{3s}\cdot\frac{1}{s}, \quad \sigma > 0$$

(iii) $L\{\delta(t-5)\}$



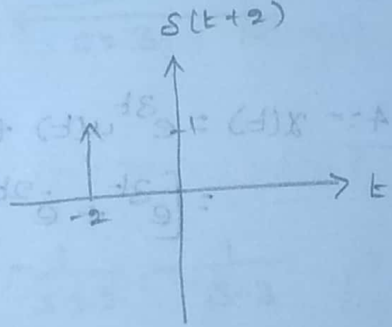
$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t-5)\, e^{-st}\, dt$$

$$= 1 \cdot e^{-st} /_{t=5}$$

$$= e^{-5s}$$ entire s plane

(iv) $L\{\delta(t+2)\}$



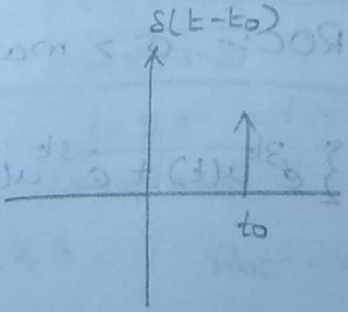
$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t+2)\, e^{-st}\, dt$$

$$= 1 \cdot e^{-st} /_{t=-2}$$

$$= e^{2s}$$

(v) $L\{\delta(t-t_0)\}$



$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-st}\, dt$$

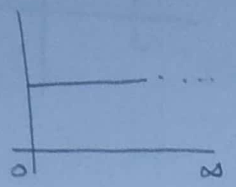
$$= 1 \cdot e^{-st} /_{t=t_0}$$

$$= e^{-t_0 s}$$

1) Infinity duration and positive $\overset{\Rightarrow \text{Right}}{}$ side signal.

Type of signal — **ROC**

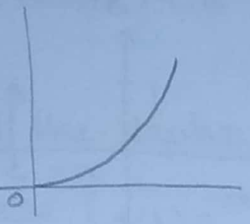
Eg-1:- $x(t) = u(t)$ — $L\{u(t)\} = \frac{1}{s-0}$, $\sigma > 0$



$s-0=0$

$s=0$

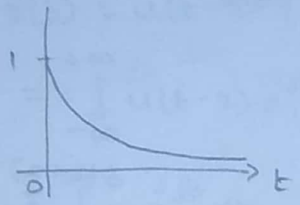
Eg-2:- $x(t) = e^{3t}u(t) \xleftrightarrow{L.T} \frac{1}{s-3}$, $\sigma > 3$



$s-3=0$

$s=3$

Eg-3:- $x(t) = e^{-2t}u(t) \xleftrightarrow{L.T} \frac{1}{s+2}$, $\sigma > -2$



$s+2=0$

$s=-2$

Eg-4:- $x(t) = e^{3t}u(t) + e^{-2t}u(t) \xleftrightarrow{L.T} \frac{1}{s+2} + \frac{1}{s-3}$

$= \left[e^{3t} + e^{-2t}\right]u(t)$

$$= \frac{s-3+s+2}{(s+2)(s-3)}$$

$$X(s) = \frac{2s-1}{(s+2)(s-3)}$$

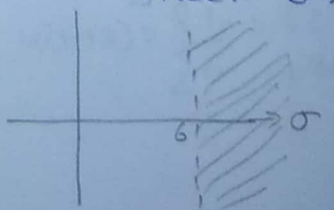
$s=-2, 3$ ROC:- $\sigma > 3$

$$\boxed{\text{ROC:- } \sigma > \max\left[\text{Roots of the denominator polynomial of } X(s)\right]}$$

5) $L\{e^{3t}u(t) + e^{-5t}u(t) + e^{6t}u(t)\} \xleftrightarrow{L.T} \frac{1}{s-3} + \frac{1}{s+5} + \frac{1}{s-6}$

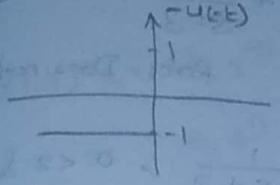
$s = 3, -5, 6.$

ROC:- $\sigma > 6$



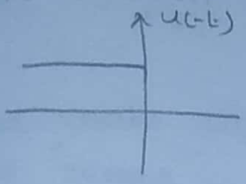
2) Infinite duration and negative (left) sided signal.

Eg-1:- $x(t) = -u(-t) \xleftrightarrow{L.T} \frac{1}{s}$, $\sigma < 0$



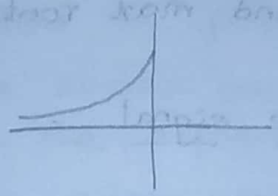
$s = 0$

$x(t) = u(-t) \xleftrightarrow{L.T} -\frac{1}{s}$, $\sigma < 0$



$s = 0$

Eg-2:- $x(t) = e^{-2t}u(-t) \xleftrightarrow{L.T} -\frac{1}{s+2}$, $\sigma < -2$



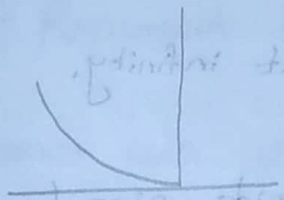
$s+2 = 0$

$s = -2$

Eg-3:- $x(t) = e^{3t}u(-t) \xleftrightarrow{L.T} -\frac{1}{s-3}$, $\sigma < 3$

$s-3 = 0$

$s = 3$



Eg-4:- $x(t) = e^{-2t}u(-t) + e^{3t}u(-t) \xleftrightarrow{L.T} -\frac{1}{s+2} - \frac{1}{s-3}$.

$$= \frac{-(s-3)-(s+2)}{(s+2)(s-3)}$$

$$X(s) = \frac{-s+3-s-2}{(s+2)(s-3)}$$

$$X(s) = \frac{1-2s}{(s+2)(s-3)}$$

$s = -2, 3$ ROC:- $\sigma < -2$

~~ROC: σ<min {Roots of the denominator polynomial}.~~

$\boxed{\text{ROC:- } \sigma < \min[\text{Roots of the denominator polynomial of } X(s)]}$

3.) Infinite duration both sided signal

Eg:- $e^{-2t}u(-t) + e^{3t}u(t) \xleftrightarrow{L.T} -\frac{1}{s+2} + \frac{1}{s-3}$

ROC:- Does not exist

$e^{2t}u(-t) + e^{-3t}u(t) \xleftrightarrow{L.T} -\frac{1}{s-2} + \frac{1}{s+3}$, $\sigma < 2$ & $\sigma >$

$= \frac{-s-3+s-2}{(s-2)(s+3)}$

ROC:- $-3 < \sigma < 2$

$X(s) = \frac{-5}{(s-2)(s+3)}$

$s = -3, 2$

ROC:- Lies between min root and max root.

4) Finite duration and positive side signal

Eg:- $x(t) = \delta(t-t_0) \xleftrightarrow{L.T} e^{-st_0}$

$\delta(t-5) \xleftrightarrow{L.T} e^{-5s}$

ROC:- Entire x plane except at infinity.

5) Finite duration and negative side signal

Eg:- $x(t) = \delta(t+t_0) \xleftrightarrow{L.T} e^{t_0 s}$

$\delta(t+5) \xleftrightarrow{L.T} e^{5s}$

ROC:- Entire x plane except $s=0$

6.) Finite duration and both sided signal

ROC:- Enitre x plane except at $\sigma=0$, $\sigma=-\infty$

Poles and Zeroes of $X(s)$:-

let

$$X(s) = \frac{a_0 s^m + a_1 s^{m-1} + \dots + a_m \rightarrow m \text{ roots}}{b_0 s^n + b_1 s^{n-1} + \dots + b_n \rightarrow n \text{ roots.}}$$

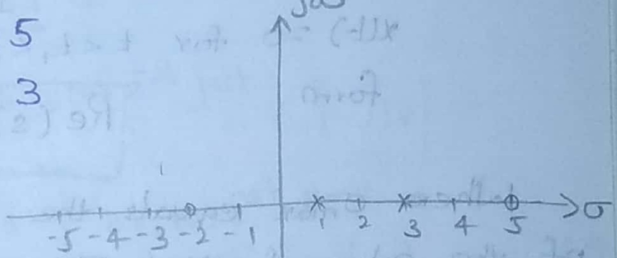
$$= \frac{a_0}{b_0} \frac{(s-z_1)(s-z_2) \dots (s-z_m)}{(s-P_1)(s-P_2) \dots (s-P_n)}$$

- * The roots of the numerator polynomial are called as "zeroes".
- * The roots of the denominator polynomial are called as "poles".
- * The zeroes are represented as small circles "o" in the s-plane.
- * The poles are represented as cross mark "x" in the s-plane.

Eg-1:- Represent poles & zeroes of $X(s) = \frac{(s-5)(s+2)}{(s-3)(s-1)}$

Zeroes are $s = -2$ & 5

poles are $s = 1$ & 3



Eg-2:- Represent poles and zeroes of $X(s) = \frac{s^2+s+1}{s^2+5s+1}$

$$s = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

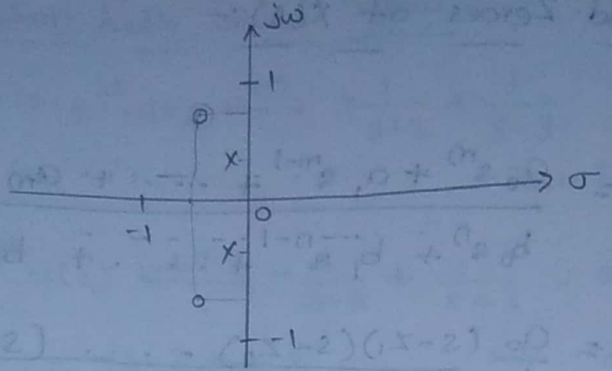
$$s^2+s+1 \Rightarrow \frac{-1 \pm \sqrt{1-4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{3}j}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}j \quad = -0.5 + 0.86602j$$

Zeroes $= -0.5 + j0.86602$

$$s^2+5s+1 \Rightarrow \frac{-5 \pm \sqrt{25-4(1)(1)}}{2(1)} = -\frac{5}{2} \pm \frac{\sqrt{21}}{2}$$

$$= 0.1 \pm 0.435j$$



The Region of Convergence :-

The range of values of the complex variables s for which the Laplace transform convergence is called the "region of convergence (ROC)."

Property 1 :- The ROC does not contain any pole.

Property 2 :- If $x(t)$ is a finite-duration signal, that is, $x(t)=0$ except in a finite interval $t_1 \le t \le t_2$ $(-\infty < t_1$ and $t_2 < \infty)$, then the ROC is the entire s-plance except possibly $s=0$ or $s=\infty$

property 3 :- If $x(t)$ is a right-sided signal, that is, $x(t)=0$ for $t < t_1 < \infty$, then the ROC is of the form

$$\boxed{Re(s) > \sigma_{max}}$$

Where σ_{max} equals the maximum real part of any of the poles of $X(s)$. Thus, the ROC is a half-plane to the right of the vertical line $Re(s) = \sigma_{max}$ in the s-plane and thus to the right of all the poles of $X(s)$

Property 4 :- If $x(t)$ is a left-sided signal, that is, $x(t)=0$ for $t > t_2 > -\infty$, then the ROC is of the form

$$\boxed{Re(s) < \sigma_{min}}$$

Where σ_{min} equals the minimum real part of any of the poles of $X(s)$. Thus, the ROC is a half-plane to the left of the vertical line $Re(s) = \sigma_{min}$ in the s-plane and thus to the left of all of the poles of $X(s)$.

Property - 5 :- If $x(t)$ is a two-sided signal, that is, $x(t)$ is an infinite duration signal that is neither right-sided nor left-sided, then the ROC is of the form

$$\boxed{\sigma_1 < Re(s) < \sigma_2}$$

Where σ_1 and σ_2 are the real parts of the two poles of $X(s)$. Thus, the ROC is a vertical strip in the s-plane between the vertical lines $Re(s) = \sigma_1$ and $Re(s) = \sigma_2$.

Note that propeity - 1 follows immediately from the definition of poles ; that is, $X(s)$ is infinite at a pole.

Properties of Laplace Transform :-

1) Linearity Property :-

Statement :- If $x_1(t) \xleftrightarrow{L.T} X_1(s)$, R_1

and $x_2(t) \xleftrightarrow{L.T} X_2(s)$, R_2

Then $a x_1(t) + b x_2(t) \xleftrightarrow{L.T} a X_1(s) + b X_2(s)$, $R_1 \cap R_2$

Proof :- $L\{a x_1(t) + b x_2(t)\} = \int_{-\infty}^{+\infty} [a x_1(t) + b x_2(t)] e^{-st} dt$.

$$= \int_{-\infty}^{+\infty} [a x_1(t) e^{-st} + b x_2(t) e^{-st}] dt.$$

$$= \int_{-\infty}^{+\infty} a x_1(t) e^{-st} dt + \int_{-\infty}^{+\infty} b x_2(t) e^{-st} dt.$$

$$= a \int_{-\infty}^{+\infty} x_1(t) e^{-st} dt + b \int_{-\infty}^{+\infty} x_2(t) e^{-st} dt.$$

$$= a X_1(s) + b X_2(s).$$

$$\therefore \boxed{a x_1(t) + b x_2(t) \xleftrightarrow{L.T} a X_1(s) + b X_2(s) \; ; \; R_1 \cap R_2}$$

2.) Time Shifting Property :-

$s = \sigma + j\omega$
$\sigma = 0$
$s = j\omega$

Statement :- If $x(t) \xleftrightarrow{L.T} X(s)$, R

then $x(t-t_0) \xleftrightarrow{F.T} e^{-j\omega t_0} X(\omega)$

$x(t-t_0) \xleftrightarrow{L.T} e^{-st_0} X(s)$, $R' = R$

Proof :- $L\{x(t-t_0)\} = \int_{-\infty}^{+\infty} [x(t-t_0)] e^{-st} dt$ — ①

– Let $t - t_0 = \lambda$

$\Rightarrow t = \lambda + t_0$

$dt = d\lambda$

Limits :- L.L $\Rightarrow t \to -\infty , \lambda \to -\infty$

U.L $\Rightarrow t \to \infty , \lambda \to \infty$

eq.-① $\Rightarrow$

$$= \int_{-\infty}^{+\infty} x(\lambda) e^{-s(\lambda + t_0)} d\lambda$$

$$= \int_{-\infty}^{+\infty} x(\lambda) e^{-s\lambda} \cdot \underset{\substack{\downarrow \\ \text{No } \lambda \text{ (constant)}}}{e^{-st_0}} d\lambda$$

$$= e^{-st_0} \int_{-\infty}^{+\infty} x(\lambda) e^{-s\lambda} d\lambda$$

$$= e^{-st_0} \cdot X(s)$$

$$\therefore \boxed{x(t-t_0) \xleftrightarrow{L.T} e^{-st_0} \cdot X(s) \quad , R' = R}$$

3.) Time Scaling Property :-

Statement :- If $x(t) \xleftrightarrow{L.T} X(s)$, R

then $x(at) \xleftrightarrow{L.T} \frac{1}{|a|} X\left(\frac{s}{a}\right)$, $R' = a.R$

Proof :- $L\{x(at)\} = \int_{-\infty}^{+\infty} x(at) e^{-st} dt$ — ①

let $at = \tau$

$t = \frac{\tau}{a}$

$dt = \frac{d\tau}{a}$

Limits :- t & τ limits are same.

$$eq\text{-}① \Rightarrow = \int_{-\infty}^{+\infty} x(\tau)\, e^{-s(\tau/a)} \frac{1}{a}\, d\tau$$

$$= \frac{1}{a}\int_{-\infty}^{+\infty} x(\tau) e^{-(\frac{s}{a})\tau}\, d\tau$$

$$= \frac{1}{a} X\left(\frac{s}{a}\right)$$

similarly if $a<0$, $x(-at) \xleftrightarrow{L.T} \frac{1}{-a} X\left(\frac{s}{a}\right)$

$$\therefore \boxed{x(at) \xleftrightarrow{L.T} \frac{1}{|a|} X\left(\frac{s}{a}\right), \quad R' = aR}$$

4) Frequency shifting property:-

Statement:- If $x(t) \xleftrightarrow{L.T} X(s)$, R

then $e^{s_0 t}\cdot x(t) \xleftrightarrow{L.T} X(s-s_0)$, $R' = R + Re\{s_0\}$

Proof:- $$L\{e^{s_0 t}\, x(t)\} = \int_{-\infty}^{+\infty} \left[e^{s_0 t}\, x(t)\right] e^{-st}\, dt$$

$$= \int_{-\infty}^{+\infty} x(t)\cdot e^{-t(s-s_0)}\, dt$$

$$= X(s-s_0)$$

$$\therefore \boxed{e^{s_0 t}\cdot x(t) \xleftrightarrow{L.T} X(s-s_0)}$$

5) Differentiation in time domain property:-

Statement:- If $x(t) \xleftrightarrow{L.T} X(s)$, R

then $\frac{d}{dt} x(t) \xleftrightarrow{L.T} s\cdot X(s)$, $R' \supseteq R$

Proof:- The inverse laplace transform is

$$\boxed{x(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} X(s)\, e^{st}\, ds} \quad — ①$$

diff w.r.t 't' on b.s.

$$\frac{d}{dt} x(t) = \frac{d}{dt}\left[\frac{1}{2\pi j}\int_{-\infty}^{\infty} X(s)\, e^{st}\, ds\right]$$

$$= \frac{1}{2\pi j}\int_{-\infty}^{+\infty} X(s)\, \frac{d}{dt} e^{st}\, ds$$

$$= \frac{1}{2\pi j}\int_{-\infty}^{+\infty} X(s)\cdot s\cdot e^{st}\, ds$$

$$\frac{d}{dt}[x(t)] = \frac{1}{2\pi j}\int_{-\infty}^{+\infty} s\cdot X(s)\, e^{st}\, ds \longleftarrow ②$$

By comparing ① & ②

$$\boxed{\frac{d}{dt}x(t) \xleftrightarrow{L\cdot T} s\cdot X(s) \quad , R' \supseteq R}$$

6.) Differentiation in frequency domain (or) S- domain property

Statement:- If $x(t) \xleftrightarrow{L\cdot T} X(s)$, R

then $t\,x(t) \xleftrightarrow{L\cdot T} -\frac{d}{ds}X(s)$, $R' = R$.

Proof:- $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$ —— ①

diff w.r.t 's' on both sides

$$\frac{d}{ds}X(s) = \frac{d}{ds}\left[\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt\right]$$

$$= \int_{-\infty}^{+\infty} x(t)\left[\frac{d}{ds}e^{-st}\right]dt$$

$$= \int_{-\infty}^{+\infty} x(t)\left[e^{-st}(-t)\right]dt.$$

$$= \int_{-\infty}^{+\infty}\left[-t\,x(t)\right]e^{-st}\, dt.$$

$$\frac{d}{ds}X(s) = \int_{-\infty}^{\infty} -t\,x(t)\, e^{-st}\, dt \longleftarrow ②$$

Comparing ① & ②

$$-t\,x(t) \xleftrightarrow{L\cdot T} \frac{d}{ds}X(s)$$

$$\boxed{t\cdot x(t) \xleftrightarrow{L\cdot T} -\frac{d}{ds}X(s)} \quad , R' = R$$

7.) Integration in time domain property :-

Statement :- If $x(t) \xleftrightarrow{L\cdot T} X(s)$, R

then $\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{L\cdot T} \frac{1}{s}X(s)$, $R' = R$

Proof:- The inverse laplace transform is

$$x(t) = \frac{1}{2\pi j}\int_{-\infty}^{+\infty} X(s)\, e^{st}\, ds.$$

Integration on both sides w.r.t t

$$\int_{-\infty}^{t} x(t)\,dt = \int_{-\infty}^{t}\left[\frac{1}{2\pi j}\int_{-\infty}^{\infty} X(s)\,e^{st}\,ds\right]dt$$

$$= \frac{1}{2\pi j}\iint_{t} X(s)\,e^{st}\,ds\,dt$$

$$= \frac{1}{2\pi j}\int_{s} X(s)\left[\int_{t} e^{st}\,dt\right]ds.$$

$$= \frac{1}{2\pi j}\int_{s} X(s)\,\frac{e^{st}}{s}\,ds.$$

$$\int x(t)\,dt = \frac{1}{2\pi j}\int_{s}\frac{X(s)}{s}\,e^{st}\,ds \quad \text{—②}$$

Comparing ① & ②

$$\int x(t)\,dt \xleftrightarrow{L.T} \frac{1}{s}X(s) \qquad R' \supset R \cap \{Re(s) > 0\}$$

8) Convolution in time domain property:-

Statement:- If $x_1(t) \xleftrightarrow{L.T} X_1(s)$, R_1

and $x_2(t) \xleftrightarrow{L.T} X_2(s)$, R_2

$$x_1(t) * x_2(t) \xleftrightarrow{L.T} X_1(s)\cdot X_2(s), \quad R' = R_1 \cap R_2$$

(or)

$$\int_{-\infty}^{\infty} x_1(\tau)\,x_2(t-\tau)\,d\tau$$

Proof:- $L\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty}[x_1(t) * x_2(t)]\,e^{-st}\,dt.$

$$= \int_{-\infty}^{\infty}\int_{\tau=-\infty}^{\infty} x_1(\tau)\cdot x_2(t-\tau)\,d\tau\,e^{-st}\,dt.$$

$$= \int_{\tau=-\infty}^{\infty} x_1(\tau)\left[\int_{t=-\infty}^{\infty} x_2(t-\tau)\,e^{-st}\,dt\right]d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x_1(\tau)\cdot X_2(s)\,e^{-s\tau}\,d\tau$$

$$= X_2(s)\cdot\int_{-\infty}^{\infty} x_1(\tau)\,e^{-s\tau}\,d\tau$$

$$= X_2(s)\cdot X_1(s).$$

$$\therefore\ x_1(t) * x_2(t) \xleftrightarrow{L.T} X_1(s)\cdot X_2(s). \qquad R' \supset R_1 \cap R_2$$

Problems

1) Find the convolution of $e^{-at}u(t)$ and $e^{-bt}u(t)$ using laplace Transform.

Sol:- $e^{-at}u(t) \xleftrightarrow{L\cdot T} \frac{1}{s+a}$, $\sigma > -a$

$e^{-bt}u(t) \xleftrightarrow{L\cdot T} \frac{1}{s+b}$, $\sigma > -b$

By using convolution in time domain property.

$$x_1(t) * x_2(t) \xleftrightarrow{L\cdot T} X_1(s)\cdot X_2(s)$$

$$L\{e^{-at}u(t) * e^{-bt}u(t)\} = \frac{1}{s+a}\cdot\frac{1}{s+b}$$

$$= \frac{1}{(s+a)(s+b)}$$

ROC:- $\sigma > -a$

if $a > b$.

2.) Find the laplace Transform of $te^{2t}u(t)$.

Sol:- By using differentiation in frequency domain property

$$tx(t) \xleftrightarrow{L\cdot T} -\frac{d}{ds}X(s)$$

let $x(t) = e^{2t}u(t)$

then $t\cdot x(t) \xleftrightarrow{L\cdot T} -\frac{d}{ds}X(s)$

Now $X(s) = L\{x(t)\} = L\{e^{2t}u(t)\} = \frac{1}{s-2}$, $\sigma > 2$

$$\therefore L\{t\cdot e^{2t}u(t)\} = -\frac{d}{ds}\left[\frac{1}{s-2}\right]$$

$$= -\left[\frac{(s-2)(0)-(1)(1)}{(s-2)^2}\right] = -\left[\frac{-1}{(s-2)^2}\right]$$

$$= \frac{1}{(s-2)^2}$$

3.) $2tu(t)$

W.K.T $tx(t) \xleftrightarrow{L\cdot T} -\frac{d}{ds}X(s)$

$2u(t) \xleftrightarrow{L\cdot T} 2\cdot\frac{1}{s}$ $\sigma > 0$

$$L[t\,x(t)] = -\frac{d}{ds}\left[\frac{2}{s}\right] = -\left[\frac{-2}{s^2}\right] = \frac{2}{s^2}$$

$$2tu(t) \xleftrightarrow{L\cdot T} \frac{2}{s^2}$$

f) $t^2 e^{5t} u(-t)$

W.K.T $\quad t\,x(t) \xleftrightarrow{L.T} -\frac{d}{ds} X(s)$

$$e^{5t}u(-t) \xleftrightarrow{L.T} \frac{-1}{s-5}, \quad \sigma < 5$$

$$L\left[t e^{5t} u(-t)\right] = -\frac{d}{ds}\left[\frac{-1}{s-5}\right] = \frac{(s-5)(0)-(1)(1)}{(s-5)^2}$$

$$= \frac{-1}{(s-5)^2}$$

$$L\left[t\left(t e^{5t} u(-t)\right)\right] = -\frac{d}{ds}\left[\frac{-1}{(s-5)^2}\right]$$

$$= \frac{(s-5)^2(0) - (1)\,2(s-5)}{(s-5)^4}$$

$$= \frac{-2\cancel{(s-5)}}{(s-5)^{\cancel{4}3}} = \frac{-2}{(s-5)^3}$$

$$t^2 e^{5t} u(-t) \xleftrightarrow{L.T} \frac{-2}{(s-5)^3}$$

Properties of the Laplace Transform

Property	Signal	Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(s) + a_2X_2(s)$	$R' \supset R_1 \cap R_2$
Time shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	$R' = R$
Shifting in s	$e^{s_0t}x(t)$	$X(s-s_0)$	$R' = R + Re(s_0)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(s)$	$R' = aR$
Time reversal	$x(-t)$	$X(-s)$	$R' = -R$
Differentiation in t	$\frac{dx(t)}{dt}$	$sX(s)$	$R' \supset R$
Differentiation in s	$-t\,x(t)$	$\frac{dX(s)}{ds}$	$R' = R$
Integration	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s)$	$R' \supset R \cap \{Re(s) > 0\}$
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	$R' \supset R_1 \cap R_2$

Some Laplace Transform Pairs

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$Re(s) > 0$
$-u(-t)$	$\frac{1}{s}$	$Re(s) < 0$
$t\,u(t)$	$1/s^2$	$Re(s) > 0$
$t^k u(t)$	$\frac{k!}{s^{k+1}}$	$Re(s) > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$Re(s) > -Re(a)$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$Re(s) < -Re(a)$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$Re(s) > -Re(a)$
$-te^{-at}u(-t)$	$\frac{1}{(s+a)^2}$	$Re(s) < -Re(a)$
$\cos\omega_0 t\,u(t)$	$\frac{s}{s^2+\omega_0^2}$	$Re(s) > 0$
$\sin\omega_0 t\,u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$Re(s) > 0$
$e^{-at}\cos\omega_0 t\,u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$Re(s) > -Re(a)$
$e^{-at}\sin\omega_0 t\,u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$Re(s) > -Re(a)$

THE INVERSE LAPLACE TRANSFORM

Inversion of the laplace transform to find the signal x(t) from its laplace transform X(s) is called the inverse Laplace transform, symbolically denoted as

$$x(t) = L^{-1}\{X(s)\}$$

A. Inversion Formula :-

There is a procedure that is applicable to all classes of transform functions that involves the evaluation of a line integral in complex s-place, that is,

$$x(t) = \frac{1}{2\pi j}\int_{s=-\infty}^{\infty} X(s)\, e^{st}\, ds \qquad x(t) = \frac{1}{2\pi j}\int_{c-j\omega}^{c+j\omega} X(s) e^{st} ds$$

In this integral, the real c is to be selected such that if the ROC of X(s) is $\sigma_1 < Re(s) < \sigma_2$, then $\sigma_1 < c < \sigma_2$. The evaluation of this inverse laplace transform integral requires an understanding of complex variable theory.

B) Use of Tables of laplace Transform Pairs :

In the second method for the inversion of X(s), we attempt to express X(s) as a sum

$$X(s) = X_1(s) + \cdots + X_n(s)$$

where $X_1(s), \ldots, X_n(s)$ are functions with known inverse transforms $x_1(t), \ldots, x_n(t)$. From the linearity property it follows that

$$x(t) = x_1(t) + \cdots + x_n(t).$$

C.) Partial - Fraction Expansion :

If X(s) is a rational function, that is, of the form.

$$X(s) = \frac{N(s)}{D(s)} = k\,\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

a simple technique based on partial - fraction expansi can be used for the inversion of X(s).

(a) When X(s) is a proper rational function, that is, when $m < n$:

1.) Simple Pole Case :

If all poles of X(s), that is, all zeros of D(s), are simple for distinct, then X(s) can be written as

$$X(s) = \frac{c_1}{s-p_1} + \cdots + \frac{c_n}{s-p_n}$$

where coefficients c_K are given by

$$c_K = (s-p_K)X(s)\Big|_{s=p_K}$$

Find the inverse laplace transform of $X(s) = \frac{s+3}{s^2+3s+2}$

Sol:- Given laplace Transform X(s) is a proper rational function.

i.e., the degree of the numerator should be less than the degree of the denominator.

$$X(s) = \frac{s+3}{(s+1)(s+2)}$$

since the poles $[s=-1 \;\&\; s=-2]$ are distinct, then

$$X(s) = \frac{s+3}{(s+1)(s+2)} = \frac{c_1}{s-(-1)} + \frac{c_2}{s-(-2)} \longrightarrow ①$$

Where $c_K = (s-p_K)X(s)\Big|_{s=p_K}$

$$c_1 = \cancel{[s-(-1)]}\,\frac{s+3}{\cancel{(s+1)}(s+2)}\Big|_{s=-1}$$

$$c_1 = \frac{s+3}{s+2}\Big|_{s=-1}$$

$$= \frac{-1+3}{-1+2} = \frac{2}{1} = 2 \qquad \boxed{\therefore c_1 = 2}$$

$$C_2 = [s-\cancel{(-2)}] \frac{s+3}{(s+1)(s+2)} \Big|_{s=-2}$$

$$= \frac{s+3}{s+1} \Big|_{s=-2}$$

$$= \frac{-2+3}{-2+1} = \frac{1}{-1} = -1$$

$$\boxed{C_2 = -1}$$

$$X(s) = \frac{2}{s-(-1)} + \frac{-1}{s-(-2)}$$

$$X(s) = 2 \cdot \frac{1}{s-(-1)} + (-1) \frac{1}{s-(-2)}$$

By using inverse laplace Transform

$$\boxed{x(t) = 2 \cdot e^{-t} u(t) - e^{-2t} u(t)}$$

Multiple Pole Case:

If D(s) has multiple roots, that is, if it contains factors of the form $(s-P_i)^r$, we say that P_i is the multiple pole of X(s) with multiplicity r. Then the expansion of X(s) will consist of terms of the form.

$$\frac{\lambda_1}{s-P_i} + \frac{\lambda_2}{(s-P_i)^2} + \dots + \frac{\lambda_r}{(s-P_i)^r}$$

Where $\lambda_{r-k} = \frac{1}{k!} \frac{d^k}{ds^k} \left[(s-P_i)^r X(s) \right] \Big|_{s=P_i}$

Find the inverse laplace transform of $X(s) = \frac{s+3}{(s+1)(s+2)^2}$

Sol:- Given Laplace transform X(s) is a proper rational function.

i.e., the degree of the numerator (1) should be less than the degree of the denominator (4)

$$X(s) = \frac{s+3}{(s+1)(s+2)^2}$$

$$P_1 = -1 \,,\; P_2 = -2$$

$$X(s) = \frac{C_1}{s-P_1} + \frac{\lambda_1}{s-P_2} + \frac{\lambda_2}{(s-P_2)^2}$$

$$= \frac{C_1}{s-(-1)} + \frac{\lambda_1}{s-(-2)} + \frac{\lambda_2}{[s-(-2)]^2}$$

$$C_1 = \cancel{[s-(-1)]}\, \frac{s+3}{\cancel{(s+1)}(s+2)^2} \Big|_{s=-1}$$

$$= \frac{-1+3}{(-1+2)^2} = \frac{2}{1} = 2$$

$$\boxed{C_1 = 2}$$

$$\lambda_{r-k} = \frac{1}{k!} \frac{d^k}{ds^k} \left[(s-P_i)^r X(s) \right] \Big|_{s=P_i} \rightarrow ①$$

put $k=0$; $r=2$ No. of times pole reated $r=2$

$$\lambda_{2-0} = \frac{1}{0!} \frac{d^0}{ds^0} \left[\cancel{(s-(-2))^2} \frac{s+3}{(s+1)\cancel{(s-(-2))^2}} \right] \Big|_{s=-2}$$

$$\lambda_2 = \frac{1}{1} \left[\frac{s+3}{s+1} \right] \Big|_{s=-2}$$

$$= \frac{-2+3}{-2+1} = \frac{1}{-1} = -1 \qquad \boxed{\lambda_2 = -1}$$

put $k=1$;

$$\lambda_{2-1} = \frac{1}{1!} \frac{d^1}{ds^1} \left[\cancel{(s-(-2))^2} \frac{s+3}{(s+1)\cancel{(s-(-2))^2}} \right] \Big|_{s=-2}$$

$$= 1 \cdot \frac{d}{ds} \left[\frac{s+3}{s+1} \right] \Big|_{s=-2}$$

$$= \frac{(s+1)(1) - (s+3)(1)}{(s+1)^2} \Big|_{s=-2}$$

$$= \frac{s+1 - s - 3}{(s+1)^2} \Big|_{s=-2} = \frac{-2}{(s+1)^2} \Big|_{s=-2}$$

$$= \frac{-2}{(-2+1)^2} = -2 \qquad \boxed{\lambda_1 = -2}$$

$$\therefore X(s) = \frac{2}{s-(-1)} + \frac{-2\cdot 1}{s-(-2)} + (-1)\cdot\frac{1}{[s-(-2)]^2}$$

$\therefore$ The Inverse laplace Transform is

$$x(t) = 2\cdot e^{-t}u(t) - 2e^{-2t}u(t) - te^{-2t}u(t).$$

(b) When $X(s)$ is an improper rational function, that is, when $m \geq n$;

If $m \geq n$, by long division we can write $X(s)$ in the form

$$X(s) = \frac{N(s)}{D(s)} = Q(s) + \frac{R(s)}{D(s)}$$

Where $N(s)$ and $D(s)$ are the numerator and Denominator polynomials in s, respectively; of $X(s)$, the quotient $Q(s)$ is a polynomial in s with degree $m-n$, and the remainder $R(s)$ is a polynomial in s with degree strictly less than n. The inverse laplace transform of $X(s)$ can then be computed by determining the inverse laplace transform of $Q(s)$ and the inverse laplace transform of $R(s)/D(s)$. Since $R(s)/D(s)$ is proper, the inverse laplace transform of $R(s)/D(s)$ can be computed by first expanding into partial fractions as given above. The inverse laplace transform of $Q(s)$ can be computed by using the transform pair

$$\frac{d^k\delta(t)}{dt^k} \longleftrightarrow s^k \qquad k = 1,2,3\ldots$$

3) Find the Inverse Laplace Transform of

$$X(s) = \frac{s^3+1}{(s+1)(s+2)(s+3)}$$

Sol:- The $X(s)$ is improper rational function because degree of Numerator = degree of denominator.

So, dividing numerator with denominator.

$$(s+1)(s+2)(s+3) = (s^2+3s+2)(s+3)$$

$$= s^3+3s^2+ 3s^2+9s+2s+6$$

$$= s^3+6s^2+11s+6.$$

$$s^3+6s^2+11s+6 \;\big)\; \cancel{s^3}+1 \;\big(\, 1$$

$$\cancel{s^3}+6s^2+11s+6$$

$$-\quad -\quad -\quad -$$

$$\overline{-6s^2-11s-5}$$

$$\therefore X(s) = 1 + \underbrace{\frac{-6s^2-11s-5}{(s+1)(s+2)(s+3)}}_{} \rightarrow F(s)$$

$$F(s) = \frac{-6s^2-11s-5}{(s+1)(s+2)(s+3)} \;\; \cancel{= \frac{C_1}{s+1} + \frac{C_2}{s+2}}$$

$$\frac{-6s^2-11s-5}{(s+1)(s+2)(s+3)} = \frac{C_1}{s-(-1)} + \frac{C_2}{s-(-2)} + \frac{C_3}{s-(-3)}$$

$$C_K = [s-P_K]\,X(s)\Big|_{s=P_K}$$

$$C_1 = [\cancel{s-(-1)}]\left[\frac{-6s^2-11s-5}{\cancel{(s+1)}(s+2)(s+3)}\right]\Bigg|_{s=-1}$$

$$C_1 = \frac{-6s^2-11s-5}{(s+2)(s+3)}\Bigg|_{s=-1}$$

$$= \frac{-6(-1)^2-11(-1)-5}{(-1+2)(-1+3)} = \frac{-6+11-5}{(1)(2)} = 0$$

$$C_1 = 0$$

$$C_2 = [s-\cancel{(-2)}]\left[\frac{-6s^2-11s-5}{(s+1)\cancel{(s+2)}(s+3)}\right]\Bigg|_{s=-2}$$

$$= \frac{-6(-2)^2-11(-2)-5}{(-2+1)(-2+3)} = \frac{-24+22-5}{(-1)(1)}$$

$$C_2 = 7$$

$$C_3 = [s-\cancel{(-3)}]\left[\frac{-6s^2-11s-5}{(s+1)(s+2)\cancel{(s+3)}}\right]\Bigg|_{s=-3}$$

$$= \frac{-6(-3)^2 - 11(-3) - 5}{(-3+1)(-3+2)} = \frac{-54+33-5}{(-2)(-1)} = \frac{-26}{2}$$

$$C_3 = -13$$

$$X(s) = 1 + 0 + \frac{7}{s+2} - 13 \cdot \frac{1}{s+3}.$$

The Inverse laplace Transform is

$$x(t) = \delta(t) + 7 \cdot e^{-2t} u(t) - 13 \cdot e^{-3t} u(t).$$

The System Function :-

The output $y(t)$ of a continuous-time LTI system equals the convolution of the input $x(t)$ with the impulse response $h(t)$; that is

$$y(t) = x(t) * h(t) \quad \text{— } 1$$

Applying the convolution property, we obtain

$$Y(s) = X(s) \cdot H(s) \quad \text{— } 2$$

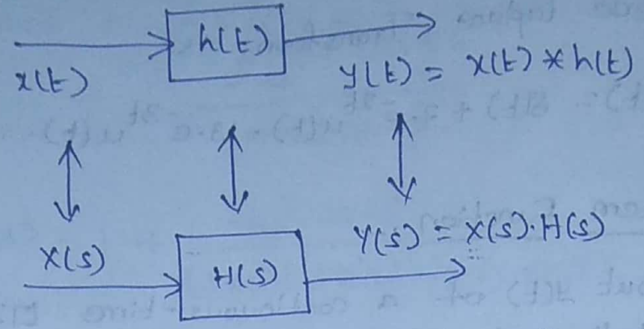
Where $Y(s)$, $X(s)$, and $H(s)$ are the Laplace transforms of $y(t)$, $x(t)$ and $h(t)$ respectively. Equation ① can be expressed as

$$H(s) = \frac{Y(s)}{X(s)} \quad \text{— } ③$$

The laplace transform $H(s)$ of $h(t)$ is referred to as the system function [or the transfer function] of the system. By eq - ③, the system function $H(s)$ can also be defined as the ratio of the laplace transform of the output $y(t)$ and the input $x(t)$. The system function $H(s)$ completely characterizes the system because the impulse response $h(t)$ completely characterizes the system. The below figure (a) illustrates the relationship of eq ① & eq ②

B) Characterization of LTI system

Many properties of continuous-time LTI systems can be closely associated with the characteristics of H(s) in the s-plane and in particular with the pole locations and the ROC.



(a) Impulse response and system function

Find ILT of $X(s) = \dfrac{s^4+7s^3+5s^2+2}{(s-1)^2(s-2)^2}$

Sol: The X(s) is improper rational function because degree of numerator = degree of denominator [m ≥ n]

$$(s-1)^2(s-2)^2 = (s^2+1-2s)(s^2+4-4s)$$

$$= s^4+4s^2-4s^3+s^2+4-4s-2s^3-8s+8s^2$$

$$= s^4-6s^3+13s^2-12s+4.$$

Dividing numerator with denominator

$$\begin{array}{r|l} s^4-6s^3+13s^2-12s+4) & s^4+7s^3+5s^2+2\ (1 \\ & s^4-6s^3+13s^2-12s+4 \\ & -\quad +\quad -\quad +\quad - \\ \hline & 13s^3-8s^2+12s-2 \end{array}$$

$$X(s) = 1 + \frac{13s^3-8s^2+12s-2}{(s-1)^2(s-2)^2}$$

let $F(s) = \dfrac{13s^3-8s^2+12s-2}{(s-1)^2(s-2)^2}$

$$F(s) = \frac{\lambda_1}{(s-1)} + \frac{\lambda_2}{(s-1)^2} + \frac{\gamma_1}{(s-2)} + \frac{\gamma_2}{(s-2)^2}$$

W.K.T

$$\lambda_{r-k} = \frac{1}{k!}\frac{d^k}{ds^k}\left[(s-p_i)^r F(s)\right]\Big|_{s=p_i}$$

No. of times pole reapeated $r=2$

put $k=0$; $r=2$

$$\lambda_{2-0} = \frac{1}{0!}\frac{d^0}{ds^0}\left[\cancel{(s-1)^2}\,\frac{13s^3-8s^2+12s-2}{\cancel{(s-1)^2}(s-2)^2}\right]\Big|_{s=1}$$

$$= \frac{13-8+12-2}{(1-2)^2} = 15$$

$$\boxed{\lambda_2 = 15}$$

put $k=1$; $r=2$

$$\lambda_{2-1} = \frac{1}{1!}\frac{d^1}{ds^1}\left[\cancel{(s-1)^2}\,\frac{13s^3-8s^2+12s-2}{\cancel{(s-1)^2}(s-2)^2}\right]\Big|_{s=1}$$

$$= \frac{(s-2)^2\left[39s^2-16s+12\right]-\left[13s^3-8s^2+12s-2\right]2(s-2)}{\left[(s-2)^2\right]^2}\Big|_{s=1}$$

$$\lambda_1 = \frac{(1-2)^2(39-16+12)-(13-8+12-2)\,2\,(1-2)}{(1-2)^4}$$

$$= \frac{35+30}{1} = 65$$

$$\boxed{\lambda_1 = 65}$$

$$\gamma_{r-k} = \frac{1}{k!}\frac{d^k}{ds^k}\left[(s-p_i)^r F(s)\right]\Big|_{s=p_i}$$

put $k=0$, $r=2$

$$\gamma_{2-0} = \frac{1}{0!}\frac{d^0}{ds^0}\left[\cancel{(s-2)^2}\,\frac{13s^3-8s^2+12s-2}{(s-1)^2\cancel{(s-2)^2}}\right]\Big|_{s=2}$$

$$= \frac{13(8)-8(4)+12(2)-2}{(2-1)^2} = 94$$

$$\boxed{\gamma_2 = 94}$$

put $k=1$, $r=2$

$$\gamma_{2-1} = \frac{1}{1!}\frac{d^1}{ds^1}\left[\cancel{(s-2)^2}\,\frac{13s^3-8s^2+12s-2}{(s-1)^2\cancel{(s-2)^2}}\right]\Big|_{s=2}$$

$$= \frac{(s-1)^2\left[39s^2-16s+12\right]-(13s^3-8s^2+12s-2)\,2(s-1)}{(s-1)^4}\Big|_{s=2}$$

$$= \frac{(2-1)^2\left[39(4)-16(2)+12\right]-\left[13(8)-8(4)+12(2)-2\right]\left[2(2-1)\right]}{(2-1)^4}$$

$$\boxed{r_1 = -52}$$

$$F(s) = \frac{65}{(s-1)} + \frac{15}{(s-1)^2} - \frac{52}{(s-2)} + \frac{94}{(s-2)^2}$$

$$\therefore X(s) = 1 + \frac{65}{(s-1)} + \frac{15}{(s-1)^2} - \frac{52}{(s-2)} + \frac{94}{(s-2)^2}$$

The Inverse laplace Transform is

$$x(t) = \delta(t) + 65e^{t}u(t) + 15te^{t}u(t) - 52e^{2t}u(t)$$
$$+ 94te^{2t}u(t).$$

Find laplace Transform of $\cos\omega_0 t\, u(t)$

Sol:-

$$\cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\cos\omega_0 t\, u(t) = \left\{\frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]\right\}u(t)$$

$$= \frac{1}{2}\left[e^{j\omega_0 t}u(t) + e^{-j\omega_0 t}u(t)\right]$$

$$e^{j\omega_0 t}u(t) \xleftrightarrow{L\cdot T} \frac{1}{s-j\omega_0}$$

$$e^{-j\omega_0 t}u(t) \xleftrightarrow{L\cdot T} \frac{1}{s+j\omega_0}$$

$$\cos\omega_0 t\, u(t) \xleftrightarrow{L\cdot T} \frac{1}{2}\left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right]$$

$$\xleftrightarrow{L\cdot T} \frac{1}{2}\left[\frac{s+\cancel{j\omega_0}+s-\cancel{j\omega_0}}{(s^2-j^2\omega_0^2)}\right]$$

$$\cos\omega_0 t\, u(t) \xleftrightarrow{L\cdot T} \frac{s}{s^2+\omega_0^2} \qquad \sigma > 0 \qquad [\because j^2 = -1]$$

Find laplace Transform of $\sin\omega_0 t\, u(t)$

W.K.T $\quad \sin\omega_0 t = \dfrac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

$$\sin\omega_0 t\, u(t) = \left\{\frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]\right\}u(t)$$

$$= \frac{1}{2j}\left[e^{j\omega_0 t}u(t) - e^{-j\omega_0 t}u(t)\right]$$

$$e^{j\omega_0 t}u(t) \xleftrightarrow{L.T} \frac{1}{s-j\omega_0}$$

$$e^{-j\omega_0 t}u(t) \xleftrightarrow{L.T} \frac{1}{s+j\omega_0}$$

$$\sin\omega_0 t\, u(t) \xleftrightarrow{L.T} \frac{1}{2j}\left[\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right]$$

$$\xleftrightarrow{L.T} \frac{1}{2j}\left[\frac{\cancel{s}+j\omega_0 - \cancel{s}+j\omega_0}{s^2-j^2\omega_0^2}\right]$$

$$\xleftrightarrow{L.T} \frac{1}{\cancel{2j}} \frac{\cancel{2j}\omega_0}{s^2+\omega_0^2}$$

$$\sin\omega_0 t\, u(t) \xleftrightarrow{L.T} \frac{\omega_0}{s^2+\omega_0^2} \quad \sigma > 0$$

Find the Inverse Laplace transform of $X(s) = \frac{2s+4}{s^2+4s+3}$

ROC :- $-3 < \sigma < -1$

Sol:- The given $X(s)$ is a proper rational function

i.e, degree of numerator < degree of denominator

$$X(s) = \frac{2s+4}{s^2+4s+3}$$

$$s^2+4s+3 = s^2+3s+s+3$$

$$= s(s+3)+(s+3)$$

$$= (s+3)(s+1)$$

poles are $-1, -3$

Since the poles $P_1 = -1$, $P_2 = -3$ are distinct

then

$$X(s) = \frac{2s+4}{(s+1)(s+3)} = \frac{C_1}{s-(-1)} + \frac{C_2}{s-(-3)}$$

$$C_k = [s-P_k]\, X(s)\,|_{s=P_k}$$

$$C_1 = [\cancel{s-(-1)}]\, \frac{2s+4}{\cancel{(s+1)}(s+3)}\,\Big|_{s=-1}$$

$$C_1 = \frac{-2+4}{-1+3} = 1$$

$$\boxed{C_1 = 1}$$

$$c_2 = \cancel{[s-(-3)]} \frac{2s+4}{\cancel{(s+3)}(s+1)} \Big|_{s=-3}$$

$$c_2 = \frac{-6+4}{-3+1} = 1$$

$$\boxed{c_2 = 1}$$

$$X(s) = \frac{1}{s-(-1)} + \frac{1}{s-(-3)} \; ; \; -3 < \sigma < -1$$

Inverse laplace Transform is

$$x(t) = -e^{-t}u(-t) + e^{-3t}u(t)$$

$\frac{1}{s-(-1)}$ I.L.T is not $e^{-t}u(t)$

because the ROC is $-3 < \sigma < -1$ i.e.,

2) Find ILT for $X(s) = \frac{2s+4}{s^2+4s+3}$, $\sigma < -3$

$$X(s) = \frac{1}{s-(-1)} + \frac{1}{s-(-3)} \; ; \; \sigma < -3$$

I.L.T

$$x(t) = -e^{-t}u(-t) + \left[-e^{-3t}u(-t)\right]$$

3) Find ILT of $X(s) = \frac{2s+4}{s^2+4s+3}$; $\sigma > -1$

$$X(s) = \frac{1}{s-(-1)} + \frac{1}{s-(-3)}$$

The poles are -1 & -3

ROC is $\sigma > \max(-1, -3)$

$\sigma > -1$

Both signals are positive or right handed signal

$$X(s) = \frac{1}{s-(-1)} + \frac{1}{s-(-3)} \; ; \; \sigma > -1$$

I.L.T is

$$x(t) = e^{-t}u(t) + e^{-3t}u(t)$$

Note:-

If ROC is given for any problem we have to write corresponding inverse laplace transform i.e., I.L.T is always based on ROC.

4) Find I.L.T of $X(s) = \frac{s+1}{(s+1)^2+9}$; $\sigma > -1$

Sol:- Given is proper rational

W.K.T $\frac{s}{s^2+b^2} \xleftrightarrow{L.T} \cos bt\, u(t)$

$\cos bt\, u(t) \xleftrightarrow{L.T} \frac{s}{s^2+b^2}$; $\sigma > 0$

From frequency shifting property

$e^{s_0 t} x(t) \xleftrightarrow{L.T} X(s-s_0)$; $\sigma = R + Re\{s_0\}$

$e^{-t}\cos 3t\, u(t) \xleftrightarrow{L.T} \frac{s-(-1)}{(s-(-1))^2+(3)^2}$ $\quad \sigma = 0 + (-1)$

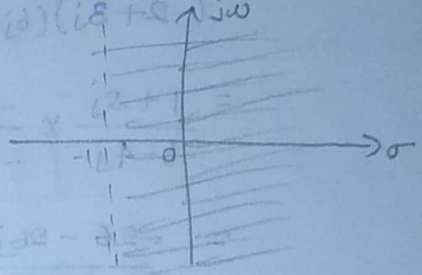
$\sigma > -1$

$s_0 = -1$

$e^{-t}\cos 3t\, u(t) \xleftrightarrow{L.T} \frac{s+1}{(s+1)^2+(3)^2}$; $\sigma > -1$

The I.L.T of $X(s)$ is

$x(t) = e^{-t}\cos 3t\, u(t)$



5) Find the Inverse Laplace Transform of $\frac{5s+13}{s(s^2+4s+13)}$, $\sigma > 0$

Sol:- The given $X(s)$ is a proper rational function

i.e., degree of numerator < degree of denominator

$$X(s) = \frac{5s+13}{s(s^2+4s+13)}$$

$s^2+4s+13$

$$s = \frac{-4 \pm \sqrt{16-4(13)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm \sqrt{(6i)^2}}{2}$$

$$= \frac{-4 \pm 6i}{2}$$

$s = -2 \pm 3i$

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s-(-2+3i)} + \frac{C_3}{s-(-2-3i)}$$

$$C_1 = (s-0)\frac{5s+13}{s(s^2+4s+13)}\Bigg|_{s=0}$$

$$= \frac{13}{0+0+13} = 1 \quad \boxed{C_1 = 1}$$

$$C_2 = \left[s-(-2+3j)\right] X(s)\Big|_{s=-2+3j}$$

$$= s-(\cancel{-2}+3j)\ \frac{5s+13}{s\left[s-(\cancel{-2+3j})\right]\left[s-(-2-3j)\right]}\Bigg|_{s=-2+3j}$$

$$= \frac{5s+13}{s\left[s-(-2-3j)\right]}\Bigg|_{s=-2+3j} = \frac{5(-2+3j)+13}{(-2+3j)\left[-\cancel{2}+3j+\cancel{2}+3j\right]}$$

$$= \frac{-10+15j+13}{(-2+3j)(6j)} = \frac{3+15j}{-12j-18} = \frac{1+5j}{-6-4j}$$

$$= \frac{1+5j}{-6-4j} \times \frac{-6+4j}{-6+4j} = \frac{-6+4j-30j-20}{36-16j^2}$$

$$= \frac{-26-26j}{52} = -\frac{1}{2}-\frac{1}{2}j$$

$$\boxed{C_2 = -\frac{1}{2}-\frac{1}{2}j}$$

$$\boxed{C_3 = -\frac{1}{2}+\frac{1}{2}j}$$

$$X(s) = \frac{1}{s} + \frac{-\frac{1}{2}-\frac{1}{2}j}{s-(-2+3j)} + \frac{-\frac{1}{2}+\frac{1}{2}j}{s-(-2-3j)}$$

I.L.T is

$$x(t) = u(t) + \left[-\frac{1}{2}-\frac{1}{2}j\right]e^{(-2+3j)t}u(t) + \left[-\frac{1}{2}+\frac{1}{2}j\right]e^{(-2-3j)t}$$

$$= u(t) + \left[-\frac{1}{2}-\frac{1}{2}j\right]e^{-2t}\cdot e^{3jt}u(t) +$$

$$\left[-\frac{1}{2}+\frac{1}{2}j\right]e^{-2t}\cdot e^{-3jt}u(t).$$

$$= u(t) - \frac{1}{2}e^{-2t}e^{3jt}u(t) - \frac{1}{2}je^{-2t}e^{3jt}u(t) - \frac{1}{2}e^{-2t}e^{-3jt}u(t)$$

$$+ \frac{1}{2}je^{-2t}e^{-3jt}u(t).$$

$$= u(t) - \frac{1}{2}e^{-2t}u(t)\left[e^{3jt} + je^{3jt} + e^{-3jt} - je^{-3jt}\right]$$

$$= u(t) - e^{-2t}u(t)\left[\frac{e^{3jt} + e^{-3jt} + je^{3jt} - je^{-3jt}}{2}\right]$$

$$= u(t) - e^{-2t}u(t)\left[\frac{e^{3jt} + e^{-3jt}}{2} + \frac{j\cdot j\left[e^{3jt} - e^{-3jt}\right]}{2j}\right]$$

$$= u(t) - e^{-2t}u(t)\left[\cos 3t - \sin 3t\right]$$

$$x(t) = \cancel{u(t)}\left[1 - e^{-2t}(\cos 3t - \sin 3t)\right]u(t)$$

$$x(t) = \left[1 - e^{-2t}(\cos 3t - \sin 3t)\right]u(t)$$

$$X(s) = \frac{5s+13}{s(s^2+4s+13)} = \frac{5s+13}{s\left[(s+2)^2+3^2\right]}$$

$$X(s) = \frac{C_1}{s} + \frac{C_2 s + C_3}{(s+2)^2+3^2} \quad \text{— ①}$$

$$C_1 = s\cdot X(s)\Big|_{s=0} \Rightarrow \boxed{C_1 = 1}$$

from ① $\Rightarrow$

$$\frac{5s+13}{s(s^2+4s+13)} = \frac{1}{s} + \frac{C_2 s + C_3}{(s+2)^2+3^2}$$

$$\Rightarrow \frac{C_2 s + C_3}{(s+2)^2+3^2} = \frac{5s+13}{s(s^2+4s+13)} - \frac{1}{s}$$

$$= \frac{5s + \cancel{13} - s^2 - 4s - \cancel{13}}{s(s^2+4s+13)}$$

$$= \frac{-s^2+s}{s(s^2+4s+13)} = \frac{\cancel{s}(1-s)}{\cancel{s}(s^2+4s+13)}$$

$$\frac{C_2 s + C_3}{(s+2)^2+3^2} = \frac{1-s}{s^2+4s+13}$$

$$\therefore X(s) = \frac{1}{s} - \frac{s-1}{(s+2)^2+3^2}$$

$$= \frac{1}{s} - \frac{s+2-3}{(s+2)^2+3^2}$$

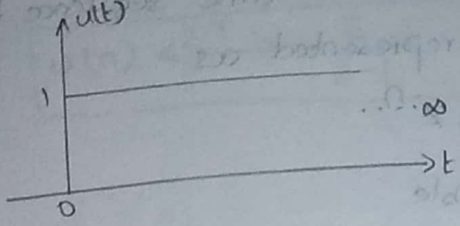
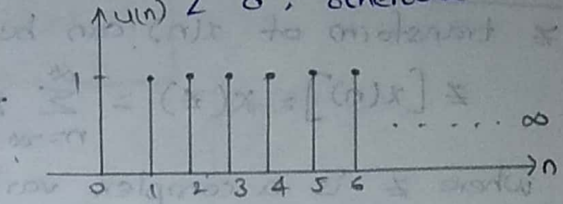
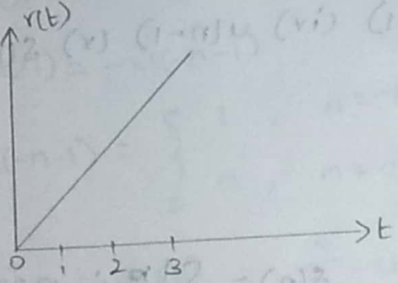
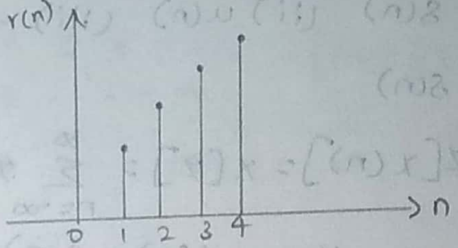
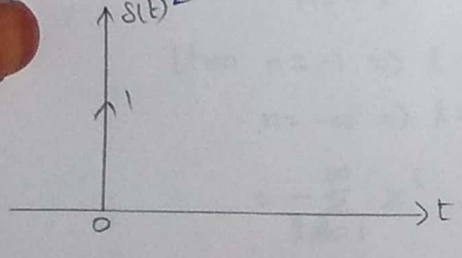
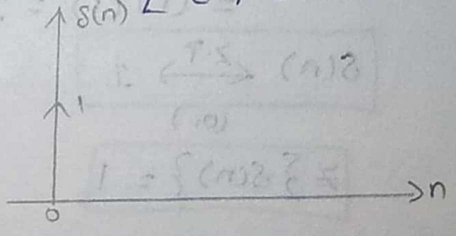
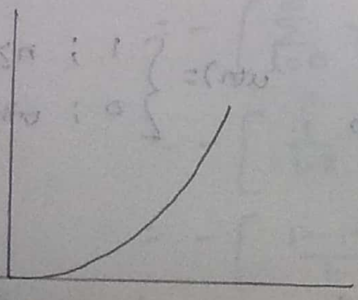
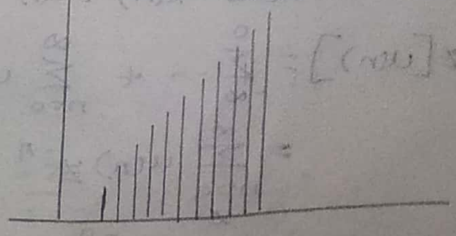
$$= \frac{1}{s} - \frac{s+2}{(s+2)^2+3^2} + \frac{3}{(s+2)^2+3^2}$$

$$= u(t) - e^{-2t}\cos 3t\, u(t) + e^{-2t}\sin 3t\, u(t)$$

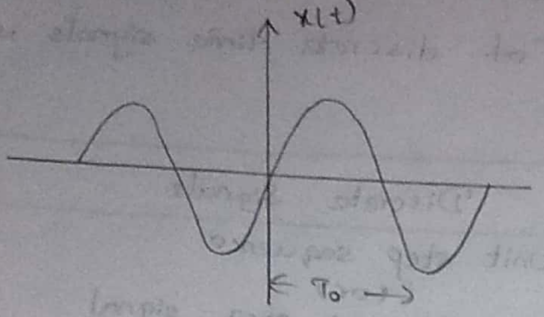
$$= \left[1 - e^{-2t}(\cos 3t - \sin 3t)\right] u(t).$$

UNIT-V Z-TRANSFORMS

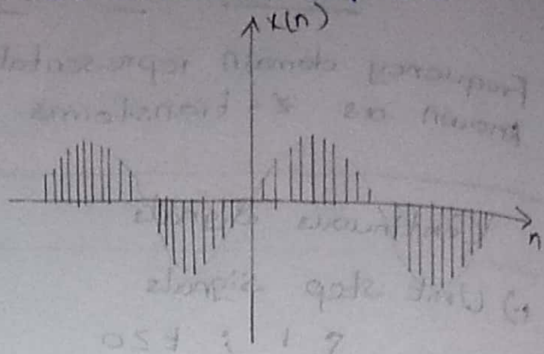
Frequency domain representation of discrete time signals is known as Z-transforms.

Continuous Signals	Discrete Signals
1) Unit step signals $u(t) = \begin{cases} 1 ; & t \geq 0 \\ 0 ; & \text{otherwise} \end{cases}$ 	1) Unit step sequence (or) Discrete unit step signal $u(n) = \begin{cases} 1 ; & n \geq 0 \\ 0 ; & \text{otherwise} \end{cases}$ 
2) Ramp Signal $r(t) = \begin{cases} t ; & t \geq 0 \\ 0 ; & t < 0 \end{cases}$ 	2) Ramp sequence (or) Discrete Ramp signal $r(n) = \begin{cases} n ; & n \geq 0 \\ 0 ; & \text{otherwise} \end{cases}$ 
3) Impulse signal $\delta(t) = \begin{cases} 1 ; & t = 0 \\ 0 ; & \text{otherwise} \end{cases}$ 	3) Impulse sequence (or) Discrete Impulse signal $\delta(n) = \begin{cases} 1 ; & n = 0 \\ 0 ; & \text{otherwise} \end{cases}$ 
4) Exponential signal 	4) Exponential sequence (or) Discrete exponential signal 

5) Sinusoidal signal



5) Sinusoidal sequence



Z-Transform:- Let x(n) be any discrete time sequence then z transform of x(n) can be represented as

$$z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n}$$

where z is a complex variable

$z = re^{(\sigma + j\omega)}$ r = magnitude.

Problems

Find z-transform of the following signals

(i) δ(n) (ii) u(n) (iii) -u(-n-1) (iv) u(n-1) (v) δ(n+2)

(i) δ(n)

$$Z[x(n)] = x[z] = \sum_{n=-\infty}^{\infty} x_n z^{-n}$$

Here $x(n) = \delta(n)$

$$\delta(n) = \begin{cases} 1 ; & n=0 \\ 0 ; & \text{otherwise} \end{cases}$$

$$z[\delta(n)] = \sum_{-\infty}^{\infty} \delta(n) z^{-n}$$

$$= \delta(n) z^{-n} \Big|_{n=0}$$

$$= (1)(1) = 1$$

$$\boxed{\delta(n) \xleftrightarrow{Z.T} 1}$$

(or)

$$\boxed{z\{\delta(n)\} = 1}$$

(ii) u(n)

$$z[x(n)] = x[z] = \sum_{n=-\infty}^{\infty} x_n z^{-n}$$

Here $x(n) = un$

$$u(n) = \begin{cases} 1 ; & n \geq 0 \\ 0 ; & \text{otherwise} \end{cases}$$

$$z[u(n)] = \sum_{-\infty}^{0} \dots + \sum_{n=0}^{\infty} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} 1 . z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$\boxed{* \ 1+a+a^2+a^3+\cdots\infty = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \ ; \ |a|<1}$$

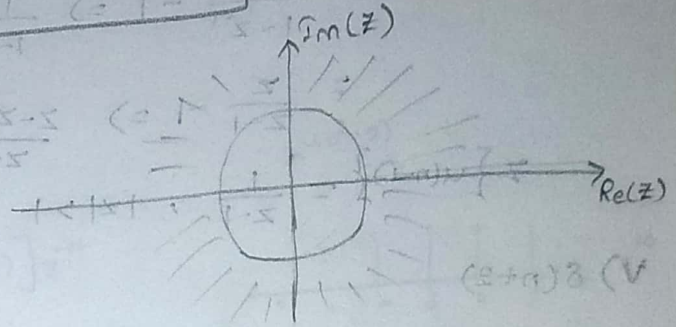
$$= \frac{1}{1-z^{-1}} \ ; \ |z^{-1}|<1$$

$$= \frac{1}{1-\frac{1}{z}} \ ; \ \left|\frac{1}{z}\right|<1$$

$$= \frac{z}{z-1} \ ; \ 1<|z|$$

$$= \frac{z}{z-1} \ ; \ |z|>1$$

$$\boxed{u(n) \overset{Z.T}{\longleftrightarrow} \frac{z}{z-1} \ ; \ |z|>1}$$



(iii) $-u(-n-1)$

$x(n) = -u(-n-1)$

$$u(-n-1) = \begin{cases} 1, & n=-1,-2,-3 \cdots -\infty \\ 0, & n=0,1,2,3 \cdots \infty \end{cases}$$

$$z\{-u(-n-1)\} = \sum_{n=-\infty}^{\infty} -u(-n-1)\, z^{-n}$$

$$= -\sum_{n=-1}^{-\infty} 1.z^{-n}$$

if $n=-l$

then $n=-1 \Rightarrow l=1$

$n=-\infty \Rightarrow l=\infty$

$$= -\sum_{l=1}^{\infty} z^{l}$$

$$= -\left[\sum_{l=1}^{\infty} z^{l} + 1 - 1\right]$$

$$= -\left[\sum_{l=0}^{\infty} z^{l} - 1\right]$$

$$= -\left[\frac{1}{1-z} - 1\right]$$

$$= -\left[\frac{1-1+z}{1-z}\right] = \frac{-z}{1-z} = \frac{-z}{-(z-1)}$$

$$z\{-u(-n-1)\} = \frac{z}{z-1} \ , \ |z|<1$$

iv) $u(n-1)$

$x(n) = u(n-1)$

$$u(n-1) = \begin{cases} 1 & ; n = 1,2,3 \cdots \infty \\ 0 & ; \text{otherwise.} \end{cases}$$

$$Z\{u(n-1)\} = \sum_{n=-\infty}^{\infty} u(n-1) z^{-n}$$

$$= \sum_{n=1}^{\infty} 1 . z^{-n} + 1 - 1$$

$$= \sum_{n=0}^{\infty} z^{-n} - 1$$

$$= \frac{1}{1-z^{-1}} - 1 \Rightarrow \frac{1}{1-\frac{1}{z}} - 1 \quad , |z^{-1}| < 1$$

$$= \frac{z}{z-1} - 1 \Rightarrow \frac{z-z+1}{z-1} \Rightarrow \frac{1}{z-1} , \left|\frac{1}{z}\right| < 1$$

$$Z\{u(n-1)\} = \frac{1}{z-1} \quad , |z| > 1$$

V) $\delta(n+2)$

$x(n) = \delta(n+2)$

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Here $x(n) = \delta(n+2)$

$$\delta(n+2) = \begin{cases} 1 & ; n = -2 \\ 0 & ; \text{otherwise} \end{cases}$$

$$Z\{\delta(n+2)\} = \sum_{n=-\infty}^{\infty} \delta(n+2) z^{-n}$$

$$= \delta(n+2) z^{-n} \Big|_{n=-2}$$

$$= 1 . z^{-(-2)} = z^{2}$$

$$\delta(n+2) \xleftrightarrow{Z.T} z^{2}.$$

(vi) $\delta(n-5)$

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Here $x(n) = \delta(n-5)$

$$\delta(n-5) = \begin{cases} 1 & ; n = 5 \\ 0 & ; \text{otherwise.} \end{cases}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n-5) z^{-n}$$

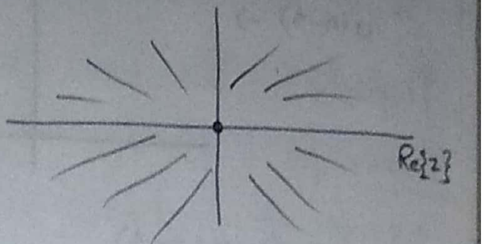
$$= \delta(n-5) z^{-n} \Big|_{n=5}$$

$$= 1 . z^{-5} = z^{-5}$$

$\delta(n-5) \xleftrightarrow{Z\cdot T} z^{-5}$

$\delta(n-5) \xleftrightarrow{Z\cdot T} \frac{1}{z^5}$

ROC :- Entire z-plane except at $z=0$



Find the Z.T of given signals

(i) $u(n-2) - u(n-5)$ (ii) $u(-n+5) - u(n-4)$ (iii) $2^n u(n)$

(iv) $-2^n u(-n-1)$.

(i) $u(n-2) - u(n-5)$

$x(n) = u(n-2) - u(n-5)$

$$X(z) = \sum_{n=-\infty}^{+\infty} [u(n-2) - u(n-5)] z^{-n}$$

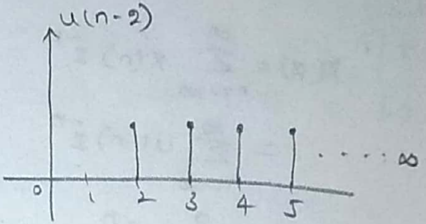
$$= \sum_{n=-\infty}^{+\infty} u(n-2) z^{-n} - \sum_{n=-\infty}^{\infty} u(n-5) z^{-n}$$

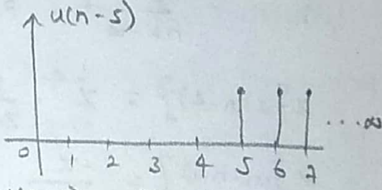
$$= \sum_{n=2}^{\infty} 1 \cdot z^{-n} - \sum_{n=5}^{\infty} 1 \cdot z^{-n}$$

$$= \frac{1}{1-z^{-1}} - 1 - z^{-1} - \left[\frac{1}{1-z^{-1}} - 1 - z^{-1} - z^{-2} - z^{-3} - z^{-4}\right]$$

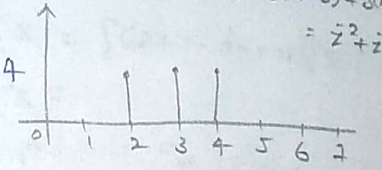
$$= \cancel{\frac{1}{1-z^{-1}}} - \cancel{1} - \cancel{z^{-1}} - \cancel{\frac{1}{1-z^{-1}}} + \cancel{1} + \cancel{z^{-1}} + z^{-2} + z^{-3} + z^{-4}$$

$X(z) = z^{-2} + z^{-3} + z^{-4}$





$u(n-2) - u(n-5) = \delta(n-2) + \delta(n-3) + \delta(n-4)$
$= z^{-2} + z^{-3} + z^{-4}$



$u(n) \xleftrightarrow{Z\cdot T} \frac{z}{z-1}$

$u(n-2) \xleftrightarrow{Z\cdot T} z^{-2} \cdot \frac{z}{z-1} = \frac{1}{z^{\cancel{2}}} \cdot \frac{\cancel{z}}{z-1} = \frac{1}{z(z-1)}$

$u(n-5) \xleftrightarrow{Z\cdot T} z^{-5} \cdot \frac{z}{z-1} = \frac{1}{z^5} \cdot \frac{z}{z-1} = \frac{1}{z^4(z-1)}$

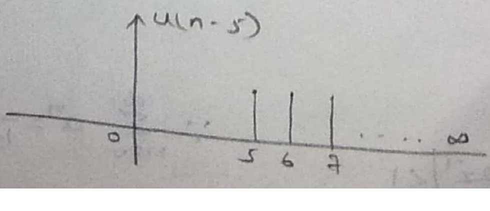
$X(z) = z^{-2} + z^{-3} + z^{-4}$

$z = re^{j\theta} = re^{j(\sigma + j\omega)}$ ROC :- Entire z-plane except at $z=0$

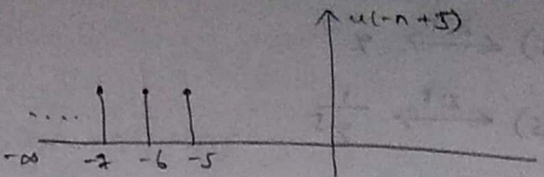
(ii) $u(-n+5) - u(n-4)$

$x(n) = u(-n+5) - u(n-4)$

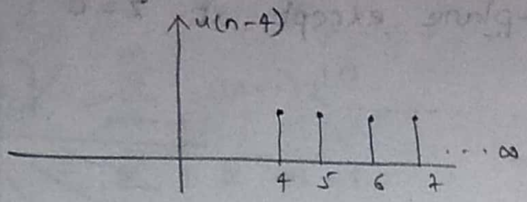
$u(n-5) \rightarrow$



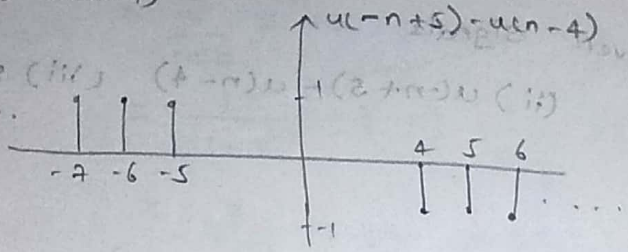
$u[-(n-5)] = u(-n+5)$



$u(n-4) \rightarrow$



$u(-n+5) - u(n-4)$



$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} u(-n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{0} z^{-n} \Rightarrow \sum_{n=0}^{\infty} z^{n} \Rightarrow \frac{1}{1-z}$$

$$Z\{u(n-4)\} = z^{-4} \cdot \frac{z}{z-1} = \frac{1}{z^3(z-1)}$$

$$Z\{u(-n-1)\} = \frac{-z}{z-1}$$

$$Z\{u(-n-1+6)\} = Z\{(-n+5)\}$$

$$= z^6\left[\frac{-z}{z-1}\right]$$

$$X(z) = -\frac{z^7}{z-1} - \frac{1}{z^3(z-1)} \Rightarrow \frac{-z^{10}-1}{z^3(z-1)}$$

(iii) $2^n u(n)$

$$x(n) = 2^n u(n)$$

$$Z.T\,[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} 2^n u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} 2^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (2z^{-1})^n$$

$$= \frac{1}{1-2z^{-1}}, \quad |2z^{-1}| < 1 \qquad \because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$= \frac{1}{1-\frac{2}{z}} \quad , \left|\frac{2}{z}\right| < 1$$

$$= \frac{z}{z-2} \quad , |2| < |z|$$

$$\boxed{2^n u(n) \xleftrightarrow{Z.T} \frac{z}{z-2} \quad , \quad |z| > 2}$$

(iv) $-2^n u(-n-1)$

$$x(n) = -2^n u(-n-1)$$

$$Z.T[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -2^n u(-n-1) z^{-n}$$

$$= -\sum_{n=-1}^{-\infty} 2^n z^{-n}$$

if $n = -l$

$n = -1 \Rightarrow l = 1$

$n = -\infty \Rightarrow l = \infty$

$$= -\sum_{l=1}^{\infty} 2^{-l} z^{l}$$

$$= -\sum_{l=1}^{\infty} (2^{-1} z)^{l}$$

$$= -\left[\sum_{l=0}^{\infty} (2^{-1} z)^{l} - 1\right]$$

$$= -\left[\frac{1}{1-2^{-1}z} - 1\right] \Rightarrow -\frac{1}{1-\frac{z}{2}} + 1 \quad , |2^{-1}z| < 1$$

$$= \frac{-2}{2-z} + 1 \Rightarrow \frac{-\cancel{2}+\cancel{2}-z}{2-z} \Rightarrow \frac{-z}{2-z} \quad ; \left|\frac{z}{2}\right| < 1$$

$$= \frac{z}{z-2} \quad , |z| < 2$$

$$-2^n u(-n-1) \xleftrightarrow{Z.T} \frac{z}{z-2} \quad , \quad |z| < 2$$

Properties of the ROC of z- Transform :-

The ROC of $X(z)$ depends on the nature of $x[n]$. The properties of the ROC are summarized below. We assume that $X(z)$ is a rational function of z.

Property 1:- The ROC does not contain any poles.

Property 2:- If $x[n]$ is a finite sequence (that is, $x[n]=0$ except in a finite interval $N_1 \leq n \leq N_2$, where N_1 and N_2 are finite) and $X(z)$ converges for some value of z, then the ROC is the entire z-plane except possibly $z=0$ or $z=\infty$.

Property- 3:- If $x[n]$ is a right-sided sequence (that is, $x[n]=0$ for $n<N_1<\infty$) and $X(z)$ converges for some values of z, then the ROC is of the form

$$|z| > r_{max} \quad \text{or} \quad \infty > |z| > r_{max}.$$

Where r_{max} equals the largest magnitude of any of the poles of $X(z)$. Thus, the ROC is the exterior of the circle $|z| = r_{max}$ in the z-plane with the possible exception of $z=\infty$.

Property- 4:- If $x[n]$ is a left-sided sequence (that is, $x[n]=0$ for $n>N_2>-\infty$) and $X(z)$ converges for some value of z, then the ROC is of the form

$$|z| < r_{min} \quad \text{or} \quad 0<|z|<r_{min}$$

where r_{min} is the smallest magnitude of any of the poles of $X(z)$. Thus, the ROC is the interior of the circle $|z| = r_{min}$ in the z-plane with the possible exception of $z=0$

Property - 5:- If $x[n]$ is a two-sided sequence (that is, $x[n]$ is an infinite-duration sequence that is neither right-sided nor left-sided) and $X(z)$ converges for some value of z, then the ROC is of the form

$$r_1 < |z| < r_2$$

Where r_1 and r_2 are the magnitudes of the two poles of $X(z)$. Thus, the ROC is an annular ring in the z-plane between the circles $|z|=r_1$ and $|z|=r_2$ not containing any poles.

Note that Property 1 follows immediately from the definition of poles; that is, $X(z)$ is infinite at a pole.

In Textbook

1.) The ROC is a ring or disk in the z-plane centred at the origin.

2) The ROC cannot contain any poles.

3.) If $x(n)$ is an infinite duration causal sequence, the ROC is $|z| > \alpha$, i.e., it is the exterior of a circle of radius α.
If $x(n)$ is a finite duration causal sequence (right-sided sequence), the ROC is entire z-plane except at $z=0$.

4.) If $x(n)$ is an infinite duration anticausal sequence, the ROC is $|z| < \beta$, i.e. it is the interior of a circle of radius β.
If $x(n)$ is a finite duration anticausal sequence (left-sided sequence), the ROC is entire z-plane except at $z=\infty$,

5.) If $x(n)$ is a finite duration two-sided sequence, the ROC is entire z-plane except at $z=0$ and $z=\infty$.

6.) If $x(n)$ is an infinite duration, two-sided sequence, the ROC consists of a ring in the z-plane (ROC; $\alpha < |z| < \beta$) bounded on the interior and exterior by a pole, not containing any poles.

7.) The ROC of an LTI stable system contains the unit circle.

8.) The ROC must be a connected region. If $X(z)$ is rational, then its ROC is bounded by poles or extends upto infinity.

9.) $x(n) = \delta(n)$ is the only signal whose ROC is entire z-plane.

Find the z-Transform of the following

(i) $x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{4}\right)^n u(n)$

(ii) $x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{4}\right)^n u(-n-1)$

(iii) $x(n) = \left(\frac{1}{2}\right)^n u(-n-1) + \left(\frac{1}{2}\right)^n u(n)$

(iv) $x(n) = \left(\frac{1}{2}\right)^n u(-n-1) + \left(\frac{1}{4}\right)^n u(-n-1)$

(i) Sol:- $x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{4}\right)^n u(n)$

$$a^n u(n) \xleftrightarrow{Z.T} \frac{z}{z-a}, \quad |z| < a$$

$$Z\left\{\left(\tfrac{1}{2}\right)^n u(n) + \left(\tfrac{1}{4}\right)^n u(n)\right\} = Z\left\{\left(\tfrac{1}{2}\right)^n u(n)\right\} + Z\left\{\left(\tfrac{1}{4}\right)^n u(n)\right\}$$

$$= \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^n u(n) z^{-n} + \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{4}\right)^n u(n) z^{-n}$$

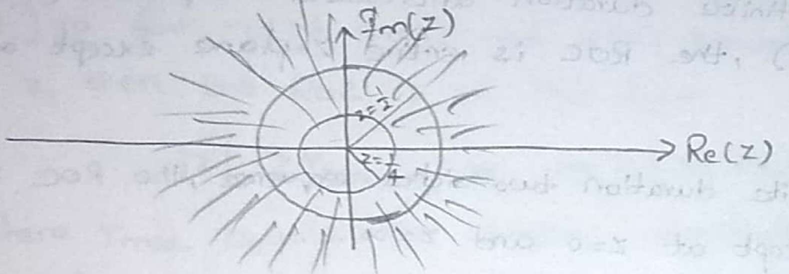
$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(\tfrac{1}{4}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(\tfrac{1}{4} z^{-1}\right)^n$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{4}z^{-1}}, \quad \left|\tfrac{1}{2}z^{-1}\right| < 1 \ \& \ \left|\tfrac{1}{4}z^{-1}\right| < 1$$

$$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z-\frac{1}{4}}, \quad \left|\tfrac{1}{2}\right| < z \ \& \ \left|\tfrac{1}{4}\right| < z$$

(or) $|z| > \frac{1}{2}$ & $|z| > \frac{1}{4}$

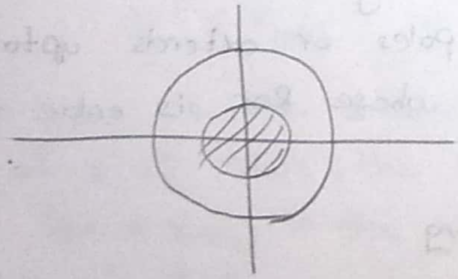


$\therefore$ ROC :- $|z| > \frac{1}{2}$

(ii) sol:- $\left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{Z.T} \frac{z}{z-\frac{1}{2}}, \quad |z| > \frac{1}{2}$

$\left(\frac{1}{4}\right)^n u(-n-1) \xleftrightarrow{Z.T} \frac{-z}{z-\frac{1}{4}}, \quad |z| < \frac{1}{4}$

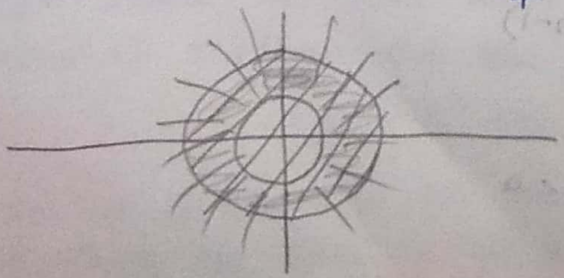
$$X(z) = Z\left\{\left(\tfrac{1}{2}\right)^n u(n) + \left(\tfrac{1}{4}\right)^n u(-n-1)\right\} = \frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{4}}$$



Since the ROC of both signals is not present, then Z.T does not exist.

(iii) sol:- $x(n) = \left(\frac{1}{2}\right)^n u(-n-1) + \left(\frac{1}{4}\right)^n u(n)$

$$X(z) = \frac{-z}{z-\frac{1}{2}} + \frac{z}{z-\frac{1}{4}} ; \quad |z| < \frac{1}{2} \ \& \ |z| > \frac{1}{4}$$



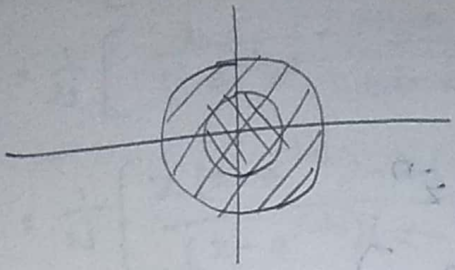
ROC:- $\frac{1}{4} \leq |z| \leq \frac{1}{2}$

iv sol:- $x(n) = \left(\frac{1}{2}\right)^n u(-n-1) + \left(\frac{1}{4}\right)^n u(-n-1)$

$X(z) = \frac{-z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{4}}$, $|z| < \frac{1}{2}$ & $|z| < \frac{1}{4}$

min of poles $\{\frac{1}{2}, \frac{1}{4}\}$

$|z| < \frac{1}{4}$



Roc:- $|z| < \frac{1}{4}$

Find the z.T. of

(i) $x(n) = \delta(n-2) + \delta(n-5)$

(ii) $x(n) = \delta(n+2) + \delta(n-5)$

(iii) $x(n) = \delta(n-2) + \delta(n+5)$

(iv) $x(n) = \delta(n+2) + \delta(n+5)$

(i) sol:- $x(n) = \delta(n-2) + \delta(n-5)$

$X(z) = z^{-2} + z^{-5}$ Roc:- Entire z-plane except at $z=0$

(ii) sol:- $x(n) = \delta(n+2) + \delta(n-5)$

$X(z) = z^{2} + z^{-5}$ Roc:- Entire z-plane except at $z=0$ & $z=\infty$

(iii) sol:- $x(n) = \delta(n-2) + \delta(n+5)$

$X(z) = z^{-2} + z^{5}$ Roc:- Entire z-plane except at $z=0$ & $z=\infty$

(iv) sol:- $x(n) = \delta(n+2) + \delta(n+5)$

$X(z) = z^{2} + z^{5}$ Roc:- Entire z-plane except at $z=\infty$.

③ Find the z.T of the following & draw the Roc of each signal.

(i) $\cos(\omega_0 n)\, u(n)$

(ii) $\sin(\omega_0 n)\, u(n)$

(iii) $r^n \cos(\omega_0 n)\, u(n)$

(iv) $r^n \sin(\omega_0 n)\, u(n)$

(i) sol:- $x(n) = \cos(\omega_0 n)\, u(n)$

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \cos(\omega_0 n)\, u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \cos(\omega_0 n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right] z^{-n}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} e^{j\omega_0 n} z^{-n} + \sum_{n=0}^{\infty} e^{-j\omega_0 n} z^{-n}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} (e^{j\omega_0} z^{-1})^n + \sum_{n=0}^{\infty} (e^{-j\omega_0} z^{-1})^n\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0} z^{-1}} + \frac{1}{1-e^{-j\omega_0} z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{Z}{Z-e^{j\omega_0}} + \frac{Z}{Z-e^{-j\omega_0}}\right]$$

$$= \frac{1}{2}\left[\frac{Z(Z-e^{-j\omega_0}) + Z(Z-e^{j\omega_0})}{(Z-e^{j\omega_0})(Z-e^{-j\omega_0})}\right]$$

$$= \frac{1}{2}\left[\frac{Z[2Z-(e^{j\omega_0}+e^{-j\omega_0})]}{Z^2 - Z(e^{j\omega_0}+e^{-j\omega_0}) + 1}\right]$$

$$= \frac{Z\left[\frac{2Z}{2} - \frac{(e^{j\omega_0}+e^{-j\omega_0})}{2}\right]}{Z^2 - 2Z\frac{(e^{j\omega_0}+e^{-j\omega_0})}{2} + 1}$$

$$= \frac{Z[Z-\cos\omega_0]}{Z^2 - 2Z\cos\omega_0 + 1}; \quad |Z|>1$$

$$\cos(\omega_0 n)\, u(n) \overset{Z.T}{\longleftrightarrow} \frac{Z(Z-\cos\omega_0)}{Z^2-2Z\cos\omega_0+1}; \quad ROC = |Z|>1.$$

(ii) sol:- $x(n) = \sin(\omega_0 n)\, u(n)$

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sin(\omega_0 n)\, u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \sin(\omega_0 n) z^{-n} \Rightarrow \sum_{n=0}^{\infty} \left[\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}\right] z^{-n}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} e^{j\omega_0 n} z^{-n} - \sum_{n=0}^{\infty} e^{-j\omega_0 n} z^{-n}\right]$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} (e^{j\omega_0} z^{-1})^n - \sum_{n=0}^{\infty} (e^{-j\omega_0} z^{-1})^n\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1-e^{j\omega_0}z^{-1}} - \frac{1}{1-e^{-j\omega_0}z^{-1}}\right] \Rightarrow \frac{1}{2j}\left[\frac{z}{z-e^{j\omega_0}} - \frac{z}{z-e^{-j\omega_0}}\right]$$

$$= \frac{1}{2j}\left[\frac{z(z-e^{-j\omega_0}) - z(z-e^{j\omega_0})}{(z-e^{j\omega_0})(z-e^{-j\omega_0})}\right]$$

$$= \frac{1}{2j}\left[\frac{\cancel{z^2} - ze^{-j\omega_0} - \cancel{z^2} + ze^{j\omega_0}}{z^2 - ze^{-j\omega_0} - ze^{j\omega_0} + 1}\right]$$

$$= \frac{1}{2j}\left[\frac{z(e^{j\omega_0} - e^{-j\omega_0})}{z^2 - z(e^{j\omega_0} + e^{-j\omega_0}) + 1}\right]$$

$$= \frac{z\left(\frac{e^{j\omega_0} - e^{-j\omega_0}}{2j}\right)}{z^2 - 2z\left(\frac{e^{j\omega_0} + e^{-j\omega_0}}{2}\right) + 1} \Rightarrow \frac{z\sin\omega_0}{z^2 - 2z\cos\omega_0 + 1}$$

$$\sin(\omega_0 n)u(n) \xleftrightarrow{Z.T} \frac{z\sin\omega_0}{z^2 - 2z\cos\omega_0 + 1} \quad ; \text{ Roc :- } |z| > 1$$

(iii) sol:- $x(n) = r^n\cos(\omega_0 n)u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x^n z^{-n} \Rightarrow \sum_{n=-\infty}^{\infty} r^n \cos(\omega_0 n)\, u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} r^n \cos(\omega_0 n)\, z^{-n} \Rightarrow \sum_{n=0}^{\infty} r^n \left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right] z^{-n}$$

$$\Rightarrow \frac{1}{2}\left[\sum_{n=0}^{\infty} (r z^{-1} e^{j\omega_0})^n + \sum_{n=0}^{\infty} (r.z^{-1} e^{-j\omega_0})^n\right]$$

$$= \frac{1}{2}\left[\frac{1}{1 - rz^{-1}e^{j\omega_0}} + \frac{1}{1 - rz^{-1}e^{-j\omega_0}}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z - re^{j\omega_0}} + \frac{z}{z - re^{-j\omega_0}}\right]$$

$$= \frac{1}{2}\left[\frac{z(z - re^{-j\omega_0}) + z(z - re^{j\omega_0})}{(z - re^{j\omega_0})(z - re^{-j\omega_0})}\right]$$

$$= \frac{1}{2}\left[\frac{z[2z - r(e^{j\omega_0} + e^{-j\omega_0})]}{z^2 - zre^{-j\omega_0} - zre^{j\omega_0} + r^2}\right]$$

$$= \frac{z\left[z - r\left(\frac{e^{j\omega_0}+e^{-j\omega_0}}{2}\right)\right]}{z^2 - 2zr\left(\frac{e^{j\omega_0}+e^{-j\omega_0}}{2}\right) + r^2} \Rightarrow \frac{z(z - r\cos\omega_0)}{z^2 - 2zr\cos\omega_0 + r^2}$$

$$r^n\cos(\omega_0 n)u(n) \xleftrightarrow{Z.T} \frac{z(z - r\cos\omega_0)}{z^2 - 2zr\cos\omega_0 + r^2}, \quad \underline{ROC:-}\ |z| > r$$

(iv) sol:- $x(n) = r^n\sin(\omega_0 n)u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} r^n\sin(\omega_0 n)u(n)z^{-n}$$

$$= \sum_{n=0}^{\infty} r^n\left[\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}\right]z^{-n}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} r^n e^{j\omega_0 n}z^{-n} - \sum_{n=0}^{\infty} r^n e^{-j\omega_0 n}z^{-n}\right]$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty}(re^{j\omega_0}z^{-1})^n - \sum_{n=0}^{\infty}(re^{-j\omega_0}z^{-1})^n\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1-re^{j\omega_0}z^{-1}} - \frac{1}{1-re^{-j\omega_0}z^{-1}}\right]$$

$$= \frac{1}{2j}\left[\frac{z}{z-re^{j\omega_0}} - \frac{z}{z-re^{-j\omega_0}}\right]$$

$$= \frac{1}{2j}\left[\frac{z(z-re^{-j\omega_0}) - z(z-re^{j\omega_0})}{(z-re^{j\omega_0})(z-re^{-j\omega_0})}\right]$$

$$= \frac{1}{2j}\left[\frac{z\left[z-re^{-j\omega_0}-z+re^{j\omega_0}\right]}{z^2-zre^{-j\omega_0}-zre^{j\omega_0}+r^2(1)}\right]$$

$$= \frac{1}{2j}\left[\frac{zr(e^{j\omega_0}-e^{-j\omega_0})}{z^2-zr(e^{j\omega_0}+e^{-j\omega_0})+r^2}\right]$$

$$= \frac{zr\frac{(e^{j\omega_0}-e^{-j\omega_0})}{2j}}{z^2-2zr\frac{(e^{j\omega_0}+e^{-j\omega_0})}{2}+r^2} \Rightarrow \frac{zr\sin\omega_0}{z^2-2zr\cos\omega_0+r^2}$$

$$r^n\sin(\omega_0 n)u(n) \xleftrightarrow{Z.T} \frac{zr\sin\omega_0}{z^2-2zr\cos\omega_0+r^2}, \quad \underline{ROC:-}\ |z| > r$$

$z\{r^n u(n)\} = ?$

$$X(z) = \sum_{n=-\infty}^{\infty} r^n u(n)z^{-n}$$

$$= \sum_{n=0}^{\infty} r^n z^{-n} \Rightarrow \sum_{n=0}^{\infty}(rz^{-1})^n$$

$$= \frac{1}{1-rz^{-1}} \Rightarrow \frac{z}{z-r}, \quad \left|\frac{r}{z}\right| < 1 \Rightarrow |r| < |z| \Rightarrow |z| > r.$$

Properties of Z-Transform :-

1) Linearity Property :-

Statement :- If $x_1(n) \xleftrightarrow{Z.T} X_1(z)$, ROC $= R_1$

$x_2(n) \xleftrightarrow{Z.T} X_2(z)$, ROC $= R_2$

then $a_1x_1(n) + a_2x_2(n) \xleftrightarrow{Z.T} a_1X_1(Z) + a_2X_2(z)$;

ROC: $R' \supset R_1 \cap R_2$

Proof :- $Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{a_1x_1(n) + a_2x_2(n)\} = \sum_{n=-\infty}^{\infty} [a_1x_1(n) + a_2x_2(n)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [a_1x_1(n) z^{-n} + a_2x_2(n) z^{-n}]$$

$$= \sum_{n=-\infty}^{\infty} a_1x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} a_2x_2(n) z^{-n}$$

$$= a_1 \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$= a_1X_1(Z) + a_2X_2(Z)$$

$$\boxed{Z\{a_1x_1(n) + a_2x_2(n)\} = a_1X_1(Z) + a_2X_2(Z)}$$ ROC :- $R' \supset R_1 \cap R_2$

2) Time Shifting Property :-

Statement :- If $x(n) \xleftrightarrow{Z.T} X(Z)$, ROC :- R

then $x(n-n_0) \xleftrightarrow{Z.T} z^{-n_0} X(Z)$, ROC :- $R \cap \{0 < |z|\}$

Proof :- $Z\{x(n-n_0)\} = \sum_{n=-\infty}^{\infty} x(n-n_0) z^{-n}$ — ①

let $n - n_0 = l$

$n = l + n_0$

$\Rightarrow n \to -\infty \Rightarrow l \to -\infty$

$n \to \infty \Rightarrow l \to \infty$

eq-① :- $$= \sum_{l=-\infty}^{+\infty} x(l) z^{-(l+n_0)}$$

$$= \sum_{l=-\infty}^{\infty} \left(x(l) z^{-n_0} \right] z^{-l}$$

$$= z^{-n_0} \sum_{l=-\infty}^{\infty} x(l) z^{-l}$$

$$= z^{-n_0} X(Z)$$

$\therefore$ $$\boxed{x(n-n_0) \xleftrightarrow{Z.T} z^{-n_0} X(Z)}$$